Analogues of centraliser subalgebras for fiat 2-categories

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Joint work with M. Mackaay
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S: a finite semigroup

$\mathcal{L}, \mathcal{R}, \mathcal{J}$: Green's relations on S (equiv. classes of left, right and two-sided ideals)

Facts

- Every simple S-module has an apex (a $\mathcal{J}$-cell). Therefore we have

$$\{\text{simple } S\text{-modules}\} = \coprod_{\text{apexes } \mathcal{J}} \{\text{simple } S\text{-modules with apex } \mathcal{J}\}.$$

- For each apex $\mathcal{J}$, there exists an idempotent $e \in \mathcal{J}$. Define $H_e := \mathcal{L}_e \cap \mathcal{R}_e$. Then H_e is a **subgroup** of S.

Theorem (Clifford-Munn-Ponizovskiĭ ∼1956)

There is a bijection between the sets:

$$\left\{ \begin{array}{c} \textit{isoclasses of simple} \\ S\textit{-modules with apex } \mathcal{J} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{isoclasses of} \\ \textit{simple } H_e\textit{-modules} \end{array} \right\}.$$

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Definition

A 2-*cat* $\mathscr{C}$ is a cat enriched over the monoidal cat **Cat** of small cats.

Example

1. **Cat** : the cat of small cats
   - objects: small cats;
   - 1-morphisms: functors;
   - 2-morphisms: natural transformations;
   - composition is the usual composition;
   - identity 1-morphisms: the identity endofunctors.
2. $\mathfrak{A}_{\Bbbk}^{f}$: the cat of all small cats equiv. to A-proj for some fin. dim. algebra A over a field $\Bbbk = \overline{\Bbbk}$.

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A 2-*functor* $\mathbf{M} : \mathscr{A} \to \mathscr{C}$ is a functor which sends objects to objects, 1-mor. to 1-mor. and 2-mor. to 2-mor. such that it intertwines the categorical structures of $\mathscr{A}$ and $\mathscr{C}$.

Remark

- 2-cats, 2-functors and 2-natural transformations form a 2-cat;
- for fixed 2-cats $\mathscr{A}$ and $\mathscr{C}$, 2-functors from $\mathscr{A}$ to $\mathscr{C}$ together with 2-natural transformations and modifications form a 2-cat.

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$\Bbbk = \overline{\Bbbk}$: an algebraically closed field

Definition

A *finitary* 2-category $\mathscr{C}$ over $\Bbbk$ is a 2-cat such that

- it has finitely many objects;
- each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is a small cat. equiv. to $A_{\mathtt{i},\mathtt{j}}$-proj, where $A_{\mathtt{i},\mathtt{j}}$ is a fin. dim. $\Bbbk$-algebra;
- all compositions are (bi)additive and $\Bbbk$-linear;
- each identity 1-morphism $\mathbb{1}_{\mathtt{i}}$ is indecomposable.

A (*weakly*) *fiat* 2-category $\mathscr{C}$ is a finitary 2-cat which has a weak involution (antiequivalence) $\star : \mathscr{C} \to \mathscr{C}^{\mathrm{op}}$ and adjunction morphisms.

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2-representations

From now on, $\mathscr{C}$ is assumed to be a finitary 2-cat.

Definition

A *finitary* 2-representation $\mathbf{M}$ of $\mathscr{C}$ is a 2-functor from $\mathscr{C}$ to $\mathfrak{A}^f_{\Bbbk}$.

Let $\mathscr{C}$-afmod denote the 2-cat of finitary 2-reps of $\mathscr{C}$.

Example

For each $\mathtt{i} \in \mathscr{C}$, the 2-rep. $\mathbf{P}_{\mathtt{i}} := \mathscr{C}(\mathtt{i}, \_) \in \mathscr{C}$-afmod is called the $\mathtt{i}$-th *principal* (or Yoneda) 2-rep.

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simple transitive 2-representations

Definition

A 2-rep. $\mathbf{M} \in \mathscr{C}$-afmod is called *transitive* if for any $X, Y \in \coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$ there exists a 1-mor. F in $\mathscr{C}$ such that Y is a direct summand of $\mathbf{M}(\mathrm{F})X$.

Definition

A 2-rep. $\mathbf{M} \in \mathscr{C}$-afmod is called *simple* (*transitive*) if $\coprod_{\mathtt{i} \in \mathscr{C}} \mathbf{M}(\mathtt{i})$ has no proper $\mathscr{C}$-invariant ideals.

Theorem (Mazorchuk-Miemietz '16)

For any 2-rep. $\mathbf{M} \in \mathscr{C}$-afmod, there exists a weak Jordan-Hölder series and its all weak composition subquotients are simple transitive.

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Main problem

Problem

Classify simple transitive 2-reps for a given finitary 2-category $\mathscr{C}$.

Cells

Definition

For indecomposable 1-mor. F, G in $\mathscr{C}$, define $\mathrm{F} \geq_L \mathrm{G}$ provided that F is isomorphic to a direct summand of $\mathrm{H} \circ \mathrm{G}$ for some 1-mor. H. Then $\geq_L$ is a preorder and we call its equivalence classes *left cells*.

Similarly one defines the *right* preorder $\geq_R$ and *right cells*, and also the *two-sided* preorder $\geq_J$ and *two-sided cells*.

Definition

A two-sided cell $\mathcal{J}$ is called *strongly regular* provided that, for any $\mathcal{L}, \mathcal{R}$ in $\mathcal{J}$, we have $|\mathcal{L} \cap \mathcal{R}| = 1$.

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Let $\mathcal{L}$ be a left cell in $\mathscr{C}$ and $\mathbf{N}$ the 2-subrep. of $\mathbf{P}_{\mathtt{i}}$ gen. by all 1-mor. in add$(\{\mathrm{F} : \mathrm{F} \geq_L \mathcal{L}\})$. Then $\mathbf{N}$ has a unique maximal $\mathscr{C}$-stable ideal $\mathbf{I}$. The quotient $\mathbf{C}_{\mathcal{L}} := \mathbf{N}/\mathbf{I}$ is called the *cell 2-rep.* of $\mathscr{C}$ associated to $\mathcal{L}$.

By definition, each cell 2-rep $\mathbf{C}_{\mathcal{L}}$ is simple transitive.

Theorem (Chan-Mazorchuk '17)

For a simple transitive 2-rep. $\mathbf{M}$ of a finitary 2-cat $\mathscr{C}$ there exists a unique maximal two-sided cell among those which do not annihilate $\mathbf{M}$.

This maximal cell is called the apex of $\mathbf{M}$.

Example

For a left cell $\mathcal{L}$ in a finitary 2-cat $\mathscr{C}$, we have apex$(\mathbf{C}_{\mathcal{L}}) = \mathcal{J}(\mathcal{L})$.

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Let $\mathcal{L}$ be a left cell in $\mathscr{C}$ and $\mathbf{N}$ the 2-subrep. of $\mathbf{P}_{\mathtt{i}}$ gen. by all 1-mor. in add($\{\mathrm{F} : \mathrm{F} \geq_L \mathcal{L}\}$). Then $\mathbf{N}$ has a unique maximal $\mathscr{C}$-stable ideal $\mathbf{I}$. The quotient $\mathbf{C}_{\mathcal{L}} := \mathbf{N}/\mathbf{I}$ is called the *cell 2-rep.* of $\mathscr{C}$ associated to $\mathcal{L}$.

By definition, each cell 2-rep $\mathbf{C}_{\mathcal{L}}$ is simple transitive.

Theorem (Chan-Mazorchuk '17)

For a simple transitive 2-rep. $\mathbf{M}$ *of a finitary 2-cat* $\mathscr{C}$ *there exists a unique maximal two-sided cell among those which do not annihilate* $\mathbf{M}$.

This maximal cell is called the apex of $\mathbf{M}$.

Example

For a left cell $\mathcal{L}$ in a finitary 2-cat $\mathscr{C}$, we have $\mathrm{apex}(\mathbf{C}_{\mathcal{L}}) = \mathcal{J}(\mathcal{L})$.

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Centralizer subalgebras

From now on $\mathscr{C}$ is assumed to be a fiat 2-cat.

WLOG: let $\mathcal{J}$ be a maximal two-sided cell in $\mathscr{C}$.

Consider the colored box $\mathcal{H} := \mathcal{L} \cap \mathcal{L}^\star$, where $\mathcal{L} \subset \mathcal{J}$,

Let $\mathscr{A}$ be the 2-full 2-subcategory of $\mathscr{C}$ gen. by all 1-mor. in $\mathcal{H}$ and the (unique) relevant identity 1-mor. $\mathbb{1}_\mathtt{i}$. Note that $\mathscr{A}$ is fiat and has at most two two-sided cells, namely, $\{\mathbb{1}_\mathtt{i}\}$ and $\mathcal{H}$.

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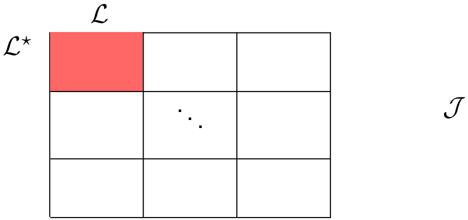

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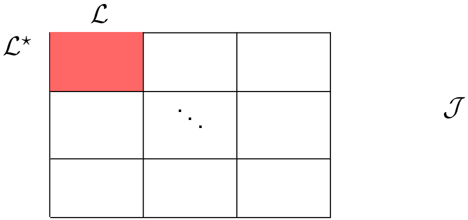

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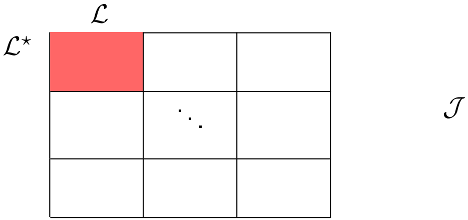

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Inspiration

S: a finite semigroup $\quad$ J: a two-sided cell

$e \in J$: an idempotent $\quad$ $H_e := L_e \cap R_e$

Theorem (Clifford-Munn-Ponizovskiĭ ∼1956)

There is a bijection between the sets:

$$\left\{ \begin{array}{c} \textit{isoclasses of simple} \\ S\textit{-modules with apex } J \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \textit{isoclasses of} \\ \textit{simple } H_e\textit{-modules} \end{array} \right\}.$$

Main Theorem

$\mathscr{C}$: a fiat 2-cat $\mathcal{J}$: a maximal two-sided cell in $\mathscr{C}$

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1. Known cases **STR** = **CR**:
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Proof: reduction to $\mathcal{H}$-cells

Construction of the map from left to right:

$\mathbf{M}$: a simple transitive 2-rep of $\mathscr{C}$ with apex $\mathcal{J}$

Then $\mathbf{M}$ is also a 2-rep of $\mathscr{A}$ via $\mathscr{A} \hookrightarrow \mathscr{C} \xrightarrow{\mathbf{M}} \mathfrak{A}_{\Bbbk}^{f}$.

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For a finitary 2-rep. **M** of $\mathscr{C}$, we also have its (diagrammatic) injective abelianization $\underline{\mathbf{M}}$, which is a 2-rep. of $\underline{\mathscr{C}}$.

C: a 1-morphism in $\underline{\mathscr{C}}$ with coalgebra structure, i.e. 2-mor. $\mathrm{C} \to \mathbb{1}_{\mathtt{i}}$ and $\mathrm{C} \to \mathrm{C} \circ \mathrm{C}$ + axioms (coalgebra 1-mor.)

$\mathrm{comod}_{\underline{\mathscr{C}}}(\mathrm{C})$: the cat of right C-comodule 1-morphisms in $\underline{\mathscr{C}}$

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Proof: reduction to $\mathcal{H}$-cells

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Theorem (Mackaay-Mazorchuk-Miemietz-Tubbenhauer '16)

Let $\mathbf{M}$ be a transitive 2-rep. of $\mathscr{C}$ and $0 \neq X \in \mathbf{M}(\mathtt{i})$ for some $\mathtt{i} \in \mathscr{C}$.

(i) *There is a coalgebra 1-mor. C_X in $\underline{\mathscr{C}}$ and an equivalence $\underline{\mathbf{M}} \simeq$ comod$_{\underline{\mathscr{C}}}(\mathrm{C}_X)$ as 2-reps of $\mathscr{C}$.*

(ii) *The above equivalence restricts to an equivalence between $\mathbf{M}$ and inj$_{\underline{\mathscr{C}}}(\mathrm{C}_X)$ as 2-reps of $\mathscr{C}$.*

Corollary (Mackaay-Mazorchuk-Miemietz-Z '18)

Let $\mathbf{M}$ be a transitive 2-rep. of $\mathscr{C}$ and $0 \neq X \in \mathbf{M}(\mathtt{i})$ for some $\mathtt{i} \in \mathscr{C}$. Consider the coalgebra 1-morphism C_X associated to $\mathbf{M}$ and X. Then

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If **N** be a simple transitive 2-rep. of $\mathscr{A}$ with apex $\mathcal{H}$, then we take $X := \bigoplus_{Y \in \mathrm{Ind}(\mathbf{N}(\mathtt{i}))/\cong} Y$ and

- Theorem MMMT $\Rightarrow$ there exists a coalgebra 1-morphism C_X in $\underline{\mathscr{A}}$ such that $\mathbf{N} \simeq \mathrm{inj}_{\underline{\mathscr{A}}}(\mathrm{C}_X)$;
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Example: $\mathscr{C}_A$

A: a connected, basic, fin. dim. **weakly symmetric** algebra over $\Bbbk = \overline{\Bbbk}$

Define $\mathscr{C}_A$ to be the 2-cat which has

- one object: a small category equiv. to A-proj;
- 1-morphisms given by functors isomorphic to $X \otimes_A \_$ where $X \in \mathrm{add}(A \oplus A \otimes_{\Bbbk} A)$;
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The 2-cat $\mathscr{C}_A$ has at most two two-sided cells. Assume that $A \ncong \Bbbk$ with a primitive decomposition $1 = e_1 + e_2 + \cdots + e_n$, then those two two-sided cells are listed as follows:

$$\mathcal{J}_0 : \boxed{{}_A A_A \otimes_A -} \quad <_J \quad \begin{array}{c|c|c|c|} & \mathcal{L}_1 & \cdots & \mathcal{L}_n \\ \hline \mathcal{R}_1 & \mathrm{F}_{11} & \cdots & \mathrm{F}_{1n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathcal{R}_n & \mathrm{F}_{n1} & \cdots & \mathrm{F}_{nn} \\ \hline \end{array} \quad : \mathcal{J}$$

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For $\mathscr{C}_A$ in the above setup, we have **STR** $=$ **CR**.

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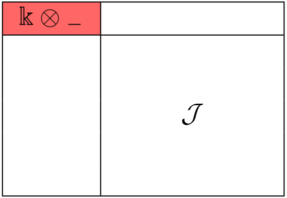
An alternative proof (use Main Theorem)

Take $B := A \times \mathbb{k}$ and consider the non-identity two-sided cell $\mathcal{J}'$ in the fiat 2-cat $\mathscr{C}_B$:

$\mathcal{J}'$:



Define the $\mathcal{H}$-cell and $\mathscr{A}$ as before. Then $\mathscr{A} = \mathscr{C}_{\mathbb{k}}$ has only one simple transitive 2-rep.

$\xRightarrow[\text{Theorem}]{\text{Main}}$ $\mathscr{C}_B$ has only one equiv. class of simple transitive 2-reps with apex $\mathcal{J}'$

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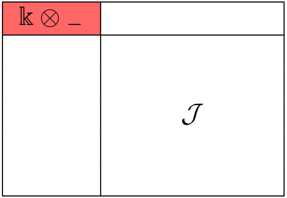

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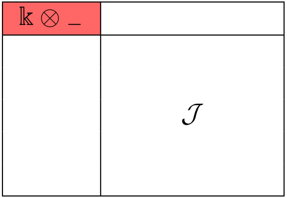

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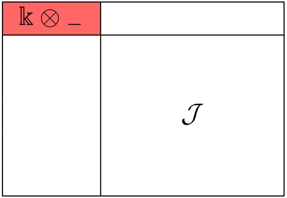

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