# GROWTH PROBLEMS IN DIAGRAM CATEGORIES

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Abstract. In the semisimple case, we derive (asymptotic) formulas for the growth rate of the number of summands in tensor powers of the generating object in diagram/interpolation categories.

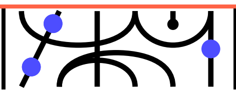
Contents

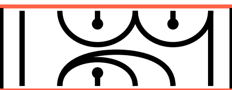
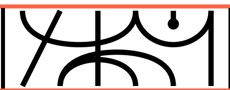
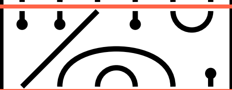
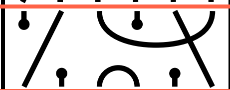
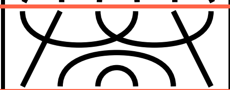
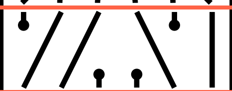
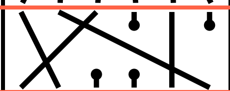
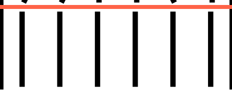
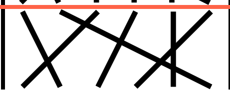


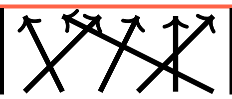
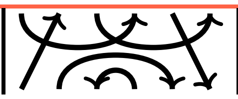
## 1. Introduction

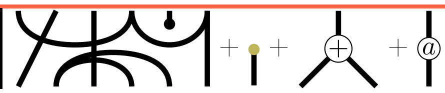
We begin with the following table, the meaning of which we will explain shortly:

(1.1)

| Symbol | Diagrams | X | $\sqrt[n]{b_n} \sim$ |
|---|---|---|---|
| $\mathcal{C}\mathrm{ob}_k$ |  | $\bullet$ | $\frac{2}{e} \cdot \frac{n}{\log 2n/k}$ |

| Symbol | Diagrams | X | $\sqrt[n]{b_n} \sim$ | Symbol | Diagrams | X | $\sqrt[n]{b_n} \sim$ |
|---|---|---|---|---|---|---|---|
| $\mathrm{p}\mathcal{P}\mathrm{a}_t$ |  | $\bullet$ | 4 | $\mathcal{P}\mathrm{a}_t$ |  | $\bullet$ | $\frac{2}{e} \cdot \frac{n}{\log 2n}$ |
| $\mathcal{M}\mathrm{o}_t$ |  | $\bullet$ | 3 | $\mathcal{R}\mathrm{o}\mathcal{B}\mathrm{r}_t$ |  | $\bullet$ | $\frac{\sqrt{2}}{\sqrt{e}} \cdot \sqrt{n}$ |
| $\mathcal{TL}_t$ |  | $\bullet$ | 2 | $\mathcal{B}\mathrm{r}_t$ |  | $\bullet$ | $\frac{\sqrt{2}}{\sqrt{e}} \cdot \sqrt{n}$ |
| $\mathrm{p}\mathcal{R}\mathrm{o}_t$ |  | $\bullet$ | 2 | $\mathcal{R}\mathrm{o}_t$ |  | $\bullet$ | $\frac{\sqrt{1}}{\sqrt{e}} \cdot \sqrt{n}$ |
| $\mathrm{p}\mathcal{S}_t$ |  | $\bullet$ | 1 | $\mathcal{S}_t$ |  | $\bullet$ | $\frac{\sqrt{1}}{\sqrt{e}} \cdot \sqrt{n}$ |

| Symbol | Diagrams | X | $\sqrt[n]{b_n} \sim$ | Symbol | Diagrams | X | $\sqrt[n]{b_n} \sim$ |
|---|---|---|---|---|---|---|---|
| $\mathrm{o}\mathcal{B}\mathrm{r}_t$ |  | $\uparrow$ | $\frac{\sqrt{1}}{\sqrt{e}} \cdot \sqrt{n}$ | $\mathrm{o}\mathcal{B}\mathrm{r}_t$ |  | $\uparrow\downarrow$ | $\frac{2}{e} \cdot n$ |

| Symbol | Diagrams | X | $\sqrt[n]{b_n} \sim$ |
|---|---|---|---|
| $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ |  for $a \in \mathbb{F}_q$ | $\bullet$ | $q^n$ |

Let $\mathbf{C}$ be an additive Krull–Schmidt monoidal category. Let $\mathtt{X} \in \mathbf{C}$ be an object of $\mathbf{C}$. We define

$$b_n = b_n^{\mathbf{C},\mathtt{X}} := \#\text{indecomposable summands in } \mathtt{X}^{\otimes n} \text{ counted with multiplicities.}$$

When $\mathbf{C}$ is semisimple, then $b_n = l_n$, the latter counting the number of simple factors in $\mathtt{X}^{\otimes n}$, and we will use this implicitly throughout.

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**Notation 1.2.** We also consider the function $\mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}, n \mapsto b_n$, which we denote by the same symbol. More generally, we identify sequences with their associated functions. $\diamond$

The function $b_n$ has been the subject of extensive study; see, for example, [CEOT24, LPRS24, Lar24, He24, HT25] for some recent work. In particular, in well-behaved categories, such as finite-dimensional representations of a group [COT24, LTV23, LTV24], one has

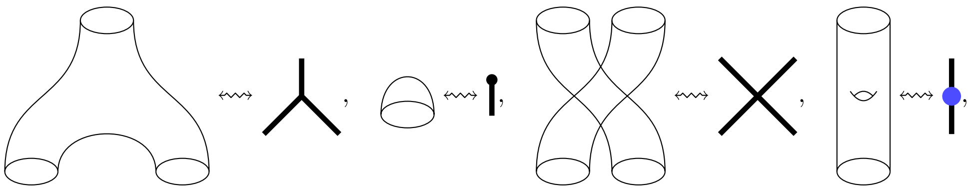
$$\lim_{n\to\infty} \sqrt[n]{b_n} \in \mathbb{R}_{\geq 1}, \quad \text{exponential growth}$$

which shows that $b_n$ grows ***exponentially***. In contrast, some still well-structured categories exhibit ***super-exponential*** growth, meaning that

$$\sqrt[n]{b_n} \text{ is unbounded}, \quad \text{superexponential growth}$$

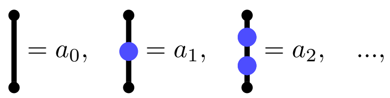
as observed in [Del07].

We study the asymptotic behavior of $\sqrt[n]{b_n}$ in the following cases, all in the semisimple situation (all parameters are generic) and over the complex numbers. Our starting point are prototypical examples of ***diagram categories***: quotients of the ***cobordism category*** $\mathcal{C}\mathrm{ob}_k$, see [KS24, KOK22] or [Koc04] for the purely topological incarnation. Here the objects are one-dimensional compact manifolds (circles) and the morphisms are two-dimensional cobordisms (pants), which we will draw using their spines with handles as dots:

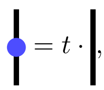



Such categories depend on the choice of a generating function as a quotient of two polynomials $p/q$, but for us only $k = \max\{\deg p + 1, \deg q\}$ plays a role. More precisely, the coefficients $a_i$ of the Taylor expansion of $p/q$ are used to evaluate closed surfaces:

(1.3)




and $k$ is the degree of a minimal polynomial of the handle. A special case for $p = 1, q = 1 - x$, so $k = 1$, is the ***partition category*** $\mathcal{P}\mathrm{a}_t$, where all handles disappear




and $t = a_0 = a_1 = \ldots \in \mathbb{C}$ (generic in this paper) is the value of floating components, which was the category studied in [Del07]. From this we also get subcategories of the partition category that are related to the classical ***diagram monoids***, defining their diagrams. These monoids are the endomorphism monoids of the respective categories for $t = 1$ (the list is taken from [KST24]):

- The ***partition monoid*** $\mathcal{P}\mathrm{a}_n$ of all diagrams of partitions of a $2n$-element set.
- The ***rook-Brauer monoid*** $\mathcal{R}\mathrm{o}\mathcal{B}\mathrm{r}_n$ consisting of all diagrams with components of size $1, 2$.
- The ***Brauer monoid*** $\mathcal{B}\mathrm{r}_n$ consisting of all diagrams with components of size 2.
- The ***rook monoid*** $\mathcal{R}\mathrm{o}_n$ consisting of all diagrams with components of size $1, 2$, and all partitions have at most one component at the bottom and at most one at the top.
- The ***symmetric group*** $\mathcal{S}_n$ consisting of all matchings with components of size 1.
- ***Planar*** versions of these: $\mathrm{p}\mathcal{P}\mathrm{a}_n$, $\mathrm{p}\mathcal{R}\mathrm{o}\mathcal{B}\mathrm{r}_n = \mathcal{M}\mathrm{o}_n$, $\mathrm{p}\mathcal{B}\mathrm{r}_n = \mathcal{TL}_n$, $\mathrm{p}\mathcal{R}\mathrm{o}_n$ and $\mathrm{p}\mathcal{S}_n \cong 1$ (the latter denotes the trivial monoid). The planar rook-Brauer monoid is also called ***Motzkin monoid***, the planar Brauer monoid is also known as ***Temperley–Lieb monoid***, and the planar symmetric group is trivial.
- Additionally, there are oriented versions of these. The one we will use is the ***oriented Brauer monoid*** $\mathrm{o}\mathcal{B}\mathrm{r}_n$, which is $\mathcal{B}\mathrm{r}_n$ with orientations.

We also consider diagram categories that do not come from monoids. Precisely, the diagram categories that interpolate the categories of finite-dimensional complex representations of $\mathrm{Rep}\big(\mathrm{GL}_n(\mathbb{F}_q)\big)$, as studied in [Kno06] (these are diagram categories by [EAH22], namely, these have partition diagrams plus extra generators).

*Remark* 1.4. When referred to as ***interpolation categories*** (potentially after taking some envelope), $\mathcal{P}\mathrm{a}_t$, $\mathcal{B}\mathrm{r}_t$ and $\mathrm{o}\mathcal{B}\mathrm{r}_t$ are often called Deligne(–Jones–Martin) categories $\mathrm{Rep}(\mathrm{S}_n)$, $\mathrm{Rep}\big(\mathrm{OSP}_t(\mathbb{C})\big)$ (or $\mathrm{Rep}\big(\mathrm{O}_t(\mathbb{C})\big)$) and $\mathrm{Rep}\big(\mathrm{GL}_t(\mathbb{C})\big)$, respectively, sometimes denote with an underline (similarly for $\mathrm{Rep}\big(\mathrm{GL}_n(\mathbb{F}_q)\big)$). Since we focus on the semisimple case, there will be no difference between the additive or abelian versions. ◇

Summarized with $\mathtt{X}$ denoting the chosen object and $\sim$ meaning ***asymptotically equal***, our results are summarized in (1.1).

Following [**KST24**], we establish these results using the Green relations and the cell structure of the associated monoids and diagram algebras, with the key ingredient being their ***sandwich cellular structure*** [**Bro55**, **TV23**, **Tub24**]. This (new) perspective simplifies the problem significantly: in most cases, the first step of counting has already been carried out in semigroup theory—albeit from a very different viewpoint—under the framework of Gelfand models [**HR15**]. The only exceptions are $\mathcal{C}\mathrm{ob}_k$ and $\mathrm{Rep}\big(\mathrm{GL}_t(\mathbb{F}_q)\big)$, which we analyze in detail. This approach yields exact formulas for $b_n$.

The remaining task—deriving asymptotics from these exact formulas—is nontrivial. However, as we show below, we have formalized the process so that it is computer-verified. For brevity, we have outsourced the computational aspects to [**GT25**], where they are freely available (at least in 2025).

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## 2. Schur–Weyl duality, sandwich cellular algebras and growth problems

For some field $\mathbb{K}$, assume that one has an additive Krull–Schmidt monoidal $\mathbb{K}$-linear category $\mathbf{C}$ with finite-dimensional hom-spaces, and an object $\mathtt{X} \in \mathbf{C}$. For simplicity, assume that $\mathbf{C}$ is semisimple. A version of Schur–Weyl duality implies that ($\mathtt{Y}$ is a simple summand of $\mathtt{X}^{\otimes n}$ that appears with multiplicity $m > 0$ if and only if the semisimple algebra $A_n = \mathrm{End}_{\mathbf{C}}(\mathtt{X}^{\otimes n})$ has a simple representation $L_{\mathtt{Y}}$ of dimension $\dim_{\mathbb{K}} L_{\mathtt{Y}} = m$), and there is a bijection between such $\mathtt{Y}$ and $L_{\mathtt{Y}}$. In particular, we have:

**Lemma 2.1.** *In the above setting, $A_n$ is semisimple and*

$$b_n = \sum_L \dim_{\mathbb{K}} L, \quad (\textit{sum over simple } A_n\textit{-representations } L).$$

*Proof.* Directly from the above discussion, which, in turn, can be justified as in [**AST18**, Section 4C]. See also [**Erd95**, **Soe99**]. □

We will now assume familiarity with sandwich cellular algebras [**Bro55**, **TV23**, **Tub24**], or at least with some variation of it, most notably, [**FG95**].

**Lemma 2.2.** *If $A_n$ is a semisimple involutive sandwich cellular algebra with apex set $\mathcal{P}^{ap}$, bottom sets $\mathcal{B}_\lambda$ and sandwiches algebras $\mathcal{H}_\lambda$, then*

$$b_n = \sum_{\lambda \in \mathcal{P}^{ap}} \big(\#\mathcal{B}_\lambda \cdot \sum_L \dim_{\mathbb{K}} L\big), \quad (\textit{inner sum over simple } \mathcal{H}_\lambda\textit{-representations } L).$$

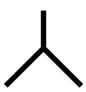
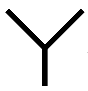
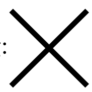
*Proof.* Directly from Lemma 2.1 and the standard theory of sandwich cellular algebras. □

We will use Lemma 2.2 throughout.

## 3. Examples

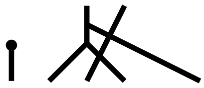
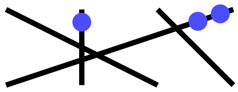
We now go through the list of example in Section 1. We will discuss $\mathcal{C}\mathrm{ob}_k$ and $\mathrm{Rep}\big(\mathrm{GL}_t(\mathbb{F}_q)\big)$ carefully. All other cases are similar to the count in $\mathcal{C}\mathrm{ob}_k$, so we just list the needed results for them.

3A. **Cobordisms.** In the notation of the introduction, let $\mathcal{C}\mathrm{ob}_\infty$ be the monoidal category with $\otimes$-generating object $\bullet$ and generating $\circ$-$\otimes$-generating morphisms

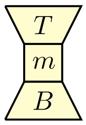
multiplication: , comultiplication: , unit: , counit: , crossing: ,

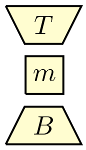
modulo the $\circ$-$\otimes$-ideal that makes the crossing a symmetry and $\bullet$ a symmetric Frobenius object with the structure maps matching the nomenclature. See [**Koc04**] for details.

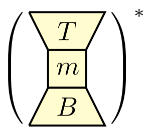
The ***diagrammatic antiinvolution*** $^*$ flips a cobordism up-side-down. We call a diagram a ***merge diagram*** if it contains only multiplications, counits and a minimal number of crossings. A ***split diagram*** is a $^*$-flipped merge diagram. A ***dotted permutation diagram*** contains only dots (=handles) and crossings. Here are some examples (flipping the left illustration gives a split diagram):

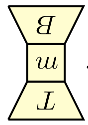
Merge diagram:  , dotted permutation diagram:  .

Abusing notation, for a field $\mathbb{K}$ we denote the $\mathbb{K}$-linear extension of the cobordism category also by $\mathcal{C}\mathrm{ob}_\infty$. Recall sandwich cellularity, the picture for $\mathcal{C}\mathrm{ob}_k$ is:

(3A.1) 

where 

a split diagram, a dotted permutation, a merge diagram. Involution: 

$=$  .

We call the existence of a spanning set with a decomposotion as above a precell structure and the respective algebras presandwich.

**Lemma 3A.2.** *The endomorphism algebras of $\mathcal{C}\mathrm{ob}_\infty$ are involutive presandwich cellular with precell structure as in (3A.1).*

*Proof.* Immediate from [**Koc04**, Section 1.4.16]. □

Now fix two polynomials $p,q\in\mathbb{K}[x]$, and consider the Taylor expansion of $p(x)/q(x)=\sum_{i=0}^{\infty}a_ix^i$, and let $k=\max\{\deg p+1,\deg q\}$. Let $\mathcal{C}\mathrm{ob}_k$ be the quotient of $\mathcal{C}\mathrm{ob}_\infty$ by the $\circ$-$\otimes$-ideal generated by (1.3). (The category $\mathcal{C}\mathrm{ob}_k$ actually depends on $p,q$ but we suppress this in the notation.)

**Example 3A.3.** For $p=1$ and $q=1-x$ we have $p(x)/q(x)=1+x+x^2+x^3+\dots$ so that all closed surfaces in $\mathcal{C}\mathrm{ob}_k$ evaluate to 1. We call the resulting category the ***partition category***. ◇

Let us denote the endomorphism monoid of $\bullet^{\otimes n}$ by $\mathcal{C}\mathrm{ob}_k(n)$. Recall the Ariki–Koike algebra (cyclotomic Hecke algebra) as defined in [**AK94**, **BM93**, **Che87**].

**Lemma 3A.4.** *Dotted permutations in $\mathcal{C}\mathrm{ob}_k(n)$ span an algebra isomorphic to the Ariki–Koike algebra $A(n,k)$ on $n$ strands with a cyclotomic relation of degree $k$ and trivial quantum parameter. Moreover, Lemma 3A.2 can be refined into a sandwich cell datum with $A(m,k)$ for $m\in\{0,\dots,n\}$ as the sandwiched algebras.*

*Proof.* This follows from [**KOK22**], e.g. the text around (11) therein, which implies that the handles satisfy a minimal polynomial of degree $k$, and the same arguments as in [**TV23**, Section 6]. □

For the next statement we assume familiarity with the usual tableaux combinatorics, see, for example, [**DJM98**, **Mat99**].

**Proposition 3A.5.** *Let $\mathbb{K}$ be of characteristic $p$.*

- **(a)** *The set of apexes of $\mathbb{K}\mathcal{C}\mathrm{ob}_k(n)$ is $\{0,\dots,n\}$.*
- **(b)** *The finite-dimensional simple $\mathbb{K}\mathcal{C}\mathrm{ob}_k(n)$-modules of apex $m$, up to equivalence, are indexed by $p$-restricted $k$-multipartitions of $m$.*
- **(c)** *The dimensions of the cell representation for a $p$-restricted $k$-multipartition $\lambda$ of $m$ is*

$$\#\{\text{merge diagrams with } m \text{ top strands}\}\cdot\#\{\text{standard tableaux of shape } \lambda\}.$$

*Moreover, if $\mathbb{K}\mathcal{C}\mathrm{ob}_k(n)$ is semisimple, then the cell representations are simple.*

*Proof.* Immediate from the standard theory of sandwich cellular algebras as, e.g., in [**Tub24**], Lemma 3A.4 and the cell structure of $A(m,k)$ as, for example, in [**DJM98**, Theorem 3.26]. □

We now need come counting lemmas.

**Lemma 3A.6.** *The number of merge diagrams from $n$ bottom strands to $m$ top strands is*

$$M_n^m=\sum_{i=m}^{n}\left\{\begin{matrix}n\\ i\end{matrix}\right\}\binom{i}{m},$$

*where the curly brackets denote the Stirling numbers of the second kind.*

*Proof.* A standard count that is independent of $k$, and therefore the same as in the partition category. Details are omitted; however, if the reader encounters difficulties, [**HR15**, Section 4] provides helpful guidance. □

Let $\mathrm{STab}(m,k)$ is the set of standard $k$-multitableaux of $m$ and let $\#\mathrm{STab}(m,k)$ denote its size. Assume from now that $\mathbb{K}=\mathbb{C}$ and that $\mathcal{C}\mathrm{ob}_k$ is semisimple.

**Lemma 3A.7.** *We have the formula*

$$b_n = \sum_{m=0}^{n} M_n^m \#\mathrm{STab}(m,k).$$

*Proof.* By Proposition 3A.5 and Lemma 3A.6. □

**Lemma 3A.8.** *$b_n$ has exponential generating function* $\exp(\frac{k}{2}\exp(2x)+\exp(x)-\frac{k+2}{2})$.

*Proof.* We first observe that

$$\#\mathrm{STab}(m,k) = k^{\lceil m/2\rceil} \sum_{i\in\mathbb{Z}_{\geq 0}} Bes(m,i) k^{\lfloor m/2\rfloor - i},$$

where we use the Bessel numbers $Bes(m,i) = m!/(i!(n-2i)!2^i)$. Thus, we get

$$b_n = \sum_{m=0}^{n} \sum_{i\in\mathbb{Z}_{\geq 0}} M_n^m k^{\lceil m/2\rceil} Bes(m,i) k^{\lfloor m/2\rfloor - i},$$

and we can use the same calculations as in [**Qua07**, Section 3] (which only uses the exponential generating functions for the Stirling and Bell numbers). □

**Lemma 3A.9.** *We have the asymptotic formula*

$$b_n \sim \frac{\left(\frac{n}{z}\right)^{n+\frac{1}{2}} \exp(\frac{k}{2})\exp(2z+1)\exp(z-n-\frac{k+2}{2})}{\sqrt{2\frac{k}{2}\exp(2z)(2z+1)+\exp(z)(z+1)}},$$

$$z = \frac{W\left(\frac{2n}{k}\right)}{2} - \frac{1}{4\left(\frac{k}{2}\right)n^{1-\frac{1}{2}}\left(W\left(\frac{2n}{k}\right)+1\right)W\left(\frac{2n}{k}\right)^{\frac{1}{2}-2} + \frac{2}{W\left(\frac{2n}{k}\right)} + 1},$$

*where $W$ is the Lambert $W$ function.*

*Proof.* Having Lemma 3A.8, the proof of this is automatized, see for example [**GT25**, **Kot22**]. This works roughly as follows. Let $f$ be the exponential generating function. One then uses Hayman's method [**Hay56**] and computes the asymptotic of $\lim_{x\to\infty} xf'(x)/f(x)$. □

**Theorem 3A.10.** *The formula in (1.1) holds.*

*Proof.* The proof is also automatized, using Lemma 3A.9 and the code on [**GT25**]. Essentially, Mathematica has a build in function for this purpose that does exact calculations. (We are referring to 'Asymptotic'; Introduced in 2020 (12.1) | Updated in 2022 (13.2).) □

3B. **Partition algebras.** For $\mathcal{P}\mathrm{a}_n$ we simply specialize $k=1$ in Section 3A. See also Example 3A.3. The corresponding sequence for $b_n$ is [**OEI23**, A002872]. However, following [**OEI23**, A002872] one gets other, slightly nastier, formulas, namely:

$$b_n \sim \left(\frac{2n}{\mathrm{W}(2n)}\right)^n \cdot \exp\left(\frac{n}{\mathrm{W}(2n)} + \left(\frac{2n}{\mathrm{W}(2n)}\right)^{\frac{1}{2}} - n - \frac{7}{4}\right) \Big/ \sqrt{1+\mathrm{W}(2n)},$$

$$\sqrt[n]{b_n} \sim \frac{2}{e}\cdot\frac{n}{\log 2n} \sqrt[\log 2n]{e}.$$

3C. **Subalgebras of partition algebras.** By an easy (and well known) diagrammatic argument we get $\mathrm{p}\mathcal{P}\mathrm{a}_n \cong \mathcal{TL}_{2n}$, we already discussed $\mathcal{P}\mathrm{a}_n$ and $\mathrm{p}\mathcal{S}_n$ is trivial, so we do not need to address these cases. Let us list the sandwich cellular bases for the remaining diagram categories:

| | | |
|---|---|---|
| | $T$ | cup and unit diagrams, |
| $\mathcal{M}\mathrm{o}_n, \mathcal{TL}_n, \mathrm{p}\mathcal{R}\mathrm{o}_n$: | $m$ | an identity, |
| | $B$ | cap and counit diagram, |

| | | |
|---|---|---|
| | $T$ | cup and unit diagrams, |
| $\mathcal{R}\mathrm{o}\mathcal{B}\mathrm{r}_n, \mathcal{B}\mathrm{r}_n, \mathcal{R}\mathrm{o}_n$: | $m$ | a permutation, |
| | $B$ | cap and counit diagram. |

For the symmetric group $\mathcal{S}_n$ the sandwich structure is trivial.

From this one gets explicit formulas, matching the ones in [**HR15**, Section 4]. Here is the list of remaining sequences:

$$\mathcal{M}\mathrm{o}_n\colon\ [\textbf{OEI23}, \text{A005773}],\quad \mathcal{R}\mathrm{o}\mathcal{B}\mathrm{r}_n\colon\ [\textbf{OEI23}, \text{A000898}],$$
$$\mathcal{TL}_n\colon\ [\textbf{OEI23}, \text{A000984}],\quad \mathcal{B}\mathrm{r}_n\colon\ [\textbf{OEI23}, \text{A047974}],$$
$$\mathrm{p}\mathcal{R}\mathrm{o}_n\colon\ [\textbf{OEI23}, \text{A000079}],\quad \mathcal{R}\mathrm{o}_n\colon\ [\textbf{OEI23}, \text{A005425}],$$
$$\mathcal{S}_n\colon\ [\textbf{OEI23}, \text{A000085}].$$

Let us just focus on $\mathcal{B}\mathrm{r}_n$; the others being similar. In this case the exponential generating function is $\exp(x^2+x)$, and Mathematica gives

$$b_n \sim 2^{\frac{n}{2}-\frac{1}{2}} \exp\left(\sqrt{\frac{n}{2}} - \frac{n}{2} - \frac{1}{8}\right) n^{\frac{n}{2}},$$

$$\sqrt[n]{b_n} \sim \frac{\sqrt{2}}{\sqrt{e}} \cdot \sqrt{n}.$$

This completes the proof.

3D. **Oriented Brauer algebras.** The left case in (1.1) is the same as for $\mathcal{S}_n$. For the right case, we observe that crossings provide isomorphisms, so we can reorder $(\uparrow\downarrow)^n$ to $n$ upwards pointing arrows followed by $n$ downwards pointing arrows. Thus, the sandwich structure is (or rather, can be arranged to be) as follows.

$\mathrm{o}\mathcal{B}\mathrm{r}_n\colon$
- $T$ cup diagrams passing the middle,
- $m$ a $\uparrow$ permutation and a $\downarrow$ permutation,
- $B$ cap diagram passing the middle.

The set of cap diagrams passing the middle can be identified with matchings of size $n-k$ of $\{1,...,n\}$ with $\{1',...,n'\}$, which has size $\binom{n}{k}^2(n-k)!$. Moreover, let $cd(k)$ denote the sum of the dimensions of simple $\mathcal{S}_k$-representations, i.e. the sequence [**OEI23**, A000085], which has exponential generating function $\exp(\frac{1}{2}x^2+x)$. A calculation, using the recursion $cd(n) = cd(n-1) + (n-1)cd(n-2)$, shows

$$cd(m+n) = \sum_{k\geq 0} k!\binom{m}{k}\binom{n}{k} cd(m-k)cd(n-k).$$

Taking everything together, we get

$$b_n = \sum_{k=0}^{n} cd(k)^2 \binom{n}{k}^2 (n-k)! = cd(2n),$$

and Mathematica proves

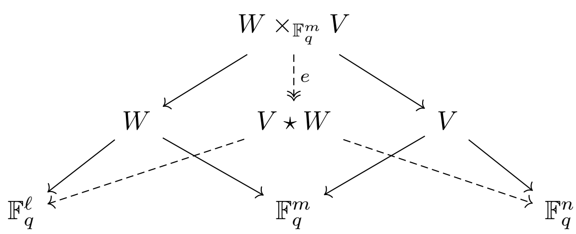
$$b_n \sim n^n 2^{n-1/2} \exp(-n + \sqrt{2n} - 1/4)(1 + 7/(24\sqrt{2n})),$$

and the result in (1.1) itself.

*Remark* 3D.1. For the lover of diagrammatics, as an alternative argument one could reorder $(\uparrow\downarrow)^n$ to $n$ upwards pointing arrows followed by $n$ downwards pointing arrows, as above, and then use that caps and cups are also invertible operations to bend the diagrams to look like $\mathcal{S}_{2n}$. $\diamond$

3E. **The general linear group over a finite field.** Throughout this subsection, we fix a prime power $q$. We start by recalling the definition of the interpolation category $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$, following [**Kno07**], and by explaining the sandwich cellular structure for endomorphism algebras in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$.

3E.1. *The definition of* $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$. For $m, n \in \mathbb{Z}_{\geq 0}$, let us write $\mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$ for the set of linear subspaces of $\mathbb{F}_q^m \oplus \mathbb{F}_q^n$. For two subspaces $V \in \mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$ and $W \in \mathrm{Gr}(\mathbb{F}_q^\ell, \mathbb{F}_q^m)$, the convolution $V \star W \in \mathrm{Gr}(\mathbb{F}_q^\ell, \mathbb{F}_q^n)$ is defined as the image in $\mathbb{F}_q^\ell \oplus \mathbb{F}_q^n$ of the pullback $W \times_{\mathbb{F}_q^m} V$, as in the following diagram.




Furthermore, we write $d(V,W) = \dim\ker(e)$ for the dimension of the kernel of the canonical epimorphism $e\colon W \times_{\mathbb{F}_q^m} V \to V \star W$.

**Definition 3E.1.** For a complex parameter $t \in \mathbb{C}$, the $\mathbb{C}$-linear category $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))_0$ has objects $\mathbb{Z}_{\geq 0}$ and

$$\mathrm{Hom}_{\mathrm{GL}_t(\mathbb{F}_q)}(m,n) = \mathbb{C}\mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$$

for $m, n \in \mathbb{Z}_{\geq 0}$, the $\mathbb{C}$-vector space with basis $\mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$. The composition of homomorphisms is defined via

$$V \circ W = t^{d(V,W)} \cdot V \star W$$

for $V \in \mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$ and $W \in \mathrm{Gr}(\mathbb{F}_q^\ell, \mathbb{F}_q^m)$, extended by bilinearity.

The category $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ is the Karoubi envelope of $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))_0$ (i.e. the idempotent completion of the additive envelope). ◇

The category $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ has a canonical $\mathbb{C}$-linear symmetric monoidal structure, with tensor unit $\mathbf{1} = 0$ and tensor product given by $m \otimes n = m + n$ and

$$V \otimes W = V \oplus W \subseteq (\mathbb{F}_q^m \oplus \mathbb{F}_q^n) \oplus (\mathbb{F}_q^{m'} \oplus \mathbb{F}_q^{n'})$$

for $V \in \mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^{m'})$ and $W \in \mathrm{Gr}(\mathbb{F}_q^n, \mathbb{F}_q^{n'})$, where the tensor product on the left hand side is taken in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ and the direct sum on the right hand side is taken in $\mathbb{F}_q$-vector spaces. Equipped with this monoidal structure, the category $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ is rigid, with evaluation and coevaluation maps both given by the diagonal subspace $\mathrm{diag}(m) = \{(v, v) \mid v \in \mathbb{F}_q^m\} \subseteq \mathbb{F}_q^{2m}$, viewed either as a homomorphism $m \otimes m \to 0$ or as a homomorphism $0 \to m \otimes m$. The dual of $V \in \mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$, viewed as a homomorphism $m \to n$ in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$, is the subspace

$$V^* = \{(v, w) \in \mathbb{F}_q^n \oplus \mathbb{F}_q^m \mid (w, v) \in V\} \in \mathrm{Gr}(\mathbb{F}_q^n, \mathbb{F}_q^m),$$

viewed as a homomorphism $n \to m$.

*Remark* 3E.2. For an $n \times m$ matrix $M$ with entries in $\mathbb{F}_q$, the graph $G(M) = \{(v, Mv) \mid v \in \mathbb{F}_q^m\}$ is a subspace of $\mathbb{F}_q^m \oplus \mathbb{F}_q^n$, and for an $m \times \ell$ matrix $M'$ with entries in $\mathbb{F}_q$, one easily checks that $G(M) \circ G(M') = G(MM')$. In particular, the group algebra $\mathbb{C}[\mathrm{GL}_n(\mathbb{F}_q)]$ is a subalgebra of the endomorphism ring $\mathrm{End}_{\mathrm{GL}_t(\mathbb{F}_q)}(n)$. ◇

3E.2. *Sandwich cellular structure.* The sandwich cellular structure of endomorphism algebras in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ relies on so-called ***core factorizations***, as explained (in a more general setting) in [**Kno07**, Section 5]. For $m \geq n \geq 0$, let us write $\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^n)$ for the set of subspaces $V$ of $\mathbb{F}_q^m \oplus \mathbb{F}_q^n$ such that the projection $V \to \mathbb{F}_q^m$ is injective and the projection $V \to \mathbb{F}_q^n$ is surjective, and similarly define $\mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$ as the set of subspaces $V$ of $\mathbb{F}_q^n \oplus \mathbb{F}_q^m$ such that the projection $V \to \mathbb{F}_q^n$ is surjective and the projection $V \to \mathbb{F}_q^m$ is injective.[1]

**Lemma 3E.3** (Core factorization)**.** *Let $m, n \in \mathbb{Z}_{\geq 0}$ and $V \in \mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$.*

- **(a)** *There is some $k \leq \min\{m, n\}$ and $V_1 \in \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$ and $V_2 \in \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$ such that $V = V_2 \circ V_1$.*
- **(b)** *For any $\ell \leq \min\{m, n\}$ and $V_1' \in \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^\ell)$ and $V_2' \in \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^\ell, \mathbb{F}_q^n)$ such that $V = V_2' \circ V_1'$, we have $\ell = k$ and there is $M \in \mathrm{GL}_k(\mathbb{F}_q)$ such that $V_1' = G(M) \circ V_1$ and $V_2 = V_2' \circ G(M)$.*

In other words, the composition of homomorphisms in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ gives rise to a bijection

$$\bigsqcup_{k \leq \min\{m,n\}} \left(\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k) \times \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)\right) / \, \mathrm{GL}_k(\mathbb{F}_q) \xrightarrow{1:1} \mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n), \tag{3E.4}$$

where $M \in \mathrm{GL}_k(\mathbb{F}_q)$ acts on $\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k) \times \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$ via $(V_1, V_2) \mapsto (G(M) \circ V_1, V_2 \circ G(M^{-1}))$. Further note that $\mathrm{GL}_k(\mathbb{F}_q)$ acts freely on $\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$ and $\mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$ via $(M, V_1) \mapsto G(M) \circ V_1$ and $(M, V_2) \mapsto V_2 \circ G(M^{-1})$, respectively. In particular, for any fixed sets of representatives $\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$ and $\mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$ for the $\mathrm{GL}_k(\mathbb{F}_q)$-orbits in $\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$ and $\mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$, respectively, we have a bijection

$$\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k) \times \mathrm{GL}_k(\mathbb{F}_q) \times \mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n) \xrightarrow{1:1} \left(\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k) \times \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)\right) / \, \mathrm{GL}_k(\mathbb{F}_q), \tag{3E.5}$$

which sends a triple $(V_1, M, V_2)$ to the $\mathrm{GL}_k(\mathbb{F}_q)$-orbit of $(G(M) \circ V_1, V_2)$. These observations essentially supply all of the data that makes up a sandwich cellular structure on the endomorphism algebras in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$, following [**Tub24**, Definition 2A.3]. Namely, for $n \in \mathbb{Z}_{\geq 0}$, the sandwich cell datum $(\mathcal{P}, (\mathcal{T}, \mathcal{B}), (\mathcal{H}_k, B_k), C)$ for the algebra $\mathcal{A} = \mathrm{End}_{\mathrm{GL}_t(\mathbb{F}_q)}(n)$ is defined as follows:

- The middle poset is $\mathcal{P} = \{k \in \mathbb{Z}_{\geq 0} \mid k \leq n\}$, endowed with the usual partial order on integers.
- The top and bottom sets are given by $\mathcal{T} = \bigsqcup_{k \leq n} \mathcal{T}(k)$ and $\mathcal{B} = \bigsqcup_{k \leq n} \mathcal{B}(k)$, with

$$\mathcal{T}(k) = \mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n), \qquad \mathcal{B}(k) = \mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k).$$

- The sandwiched algebras are the group algebras $\mathcal{H}_k = \mathbb{C}[\mathrm{GL}_k(\mathbb{F}_q)]$, with a fixed choice of basis given by $B_k = \{M \mid M \in \mathrm{GL}_k(\mathbb{F}_q)\}$.
- The map $C\colon \bigsqcup_{k \leq n} \mathcal{T}(k) \times B_k \times \mathcal{B}(k) \longrightarrow \mathcal{A}$ indexing the sandwich cellular basis is given by

$$(V_1, M, V_2) \longmapsto c^k_{V_1, M, V_2} = V_1 \circ G(M) \circ V_2.$$

We write $\mathcal{A}^{\leq k}$ and $\mathcal{A}^{<k}$ for the subspaces of $\mathcal{A}$ that are spanned by the elements $c^\ell_{V_1, M, V_2}$ with $\ell \leq k$ or $\ell < k$, respectively. The axioms $(\mathrm{AC}_1)$–$(\mathrm{AC}_3)$ from [**Tub24**, Definition 2A.3] are checked as follows:

$(\mathrm{AC}_1)$ It is clear from (3E.4) and (3E.5) that $C$ indexes the basis $\mathrm{Gr}(\mathbb{F}_q^n, \mathbb{F}_q^n)$ of

$$\mathcal{A} = \mathrm{End}_{\mathrm{GL}_t(\mathbb{F}_q)}(n) = \mathbb{C}\mathrm{Gr}(\mathbb{F}_q^n, F_q^n).$$

[1] The notation IS stands for "injective-surjective", and SI is for "surjective-injective".

(AC$_2$) Let $(V_1, M, V_2) \in \mathcal{T}(k) \times B_k \times \mathcal{B}(k)$ for some $k \leq n$, and let $W \in \mathrm{Gr}(\mathbb{F}_q^n, \mathbb{F}_q^n)$. We fix a core factorization

$$W = W_1 \circ W_2,$$

where $W_1 \in \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^\ell, \mathbb{F}_q^n)$ and $W_2 \in \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^\ell)$ for some $\ell \leq n$, and a core factorization

$$W_2 \star V_1 = X_1 \circ X_2,$$

where $X_1 \in \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^m, \mathbb{F}_q^\ell)$ and $X_2 \in \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^k, \mathbb{F}_q^m)$ for some $m \leq \min\{k, \ell\}$. Then we have

$$W \circ c^k_{V_1,M,V_2} = W_1 \circ W_2 \circ V_1 \circ G(M) \circ V_2 = t^{d(W_2,V_1)} \cdot (W_1 \circ X_1) \circ (X_2 \circ G(M) \circ V_2),$$

where $W_1 \circ X_1 \in \mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^m, \mathbb{F}_q^n)$ and $X_2 \circ G(M) \circ V_2 \in \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^m)$. If $m < k$ then it follows that

$$W \circ c^k_{V_1,M,V_2} \equiv 0 \mod \mathcal{A}^{<k},$$

and this statement is independent of $V_2$ since $m$ only depends on $W_2$ and $V_1$. If $m = k$ then there is some $M' \in \mathrm{GL}_\ell(\mathbb{F}_q)$ such that $X_2 = G(M')$, and we can further choose $X \in \mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$ and $N \in \mathrm{GL}_k(\mathbb{F}_q)$ such that

$$W_1 \circ X_1 \circ G(M') \circ G(M) = X \circ G(N).$$

Then it follows that

$$W \circ c^k_{V_1,M,V_2} = t^{d(W_2,V_1)} \cdot (X \circ G(N) \circ V_2) = t^{d(W_2,V_1)} \cdot c^k_{X,N,V_2},$$

and the scalar $t^{d(W_2,V_1)}$ is again independent of $V_2$, as required.

(AC$_3$) The cell modules are the $\mathcal{A}$-$\mathcal{H}_k$-bimodule

$$\Delta(k) = \mathbb{C}\mathrm{Gr}^{\mathrm{SI}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$$

and the $\mathcal{H}_k$-$\mathcal{A}$-bimodule

$$\nabla(k) = \mathbb{C}\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^k, \mathbb{F}_q^n).$$

The isomorphism

$$\mathcal{A}^{\leq k}/\mathcal{A}^{<k} \cong \Delta(k) \otimes_{\mathcal{H}_k} \nabla(k)$$

is straightforward from (3E.4) and (3E.5); see also the proof of Proposition 4.33 in [SS22].

In summary, the sandwich cellular structure for endomorphism algebras in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ looks as follows:

| | | |
|---|---|---|
| | $T$ | SI subspaces $\mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$, |
| $\mathrm{Rep}\big(\mathrm{GL}_t(\mathbb{F}_q)\big)$: | $m$ | the group algebra $\mathbb{C}[\mathrm{GL}_k(\mathbb{F}_q)]$, |
| | $B$ | IS subspaces $\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$. |

Another interpretation of the sandwich cellular structure will be explained in Subsection 3E.4 below, in terms of the diagrammatic presentation of $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ from [EAH22].

3E.3. *Counting direct summands.* From now on, we assume that $q$ is odd, and that $t \in \mathbb{C} \setminus \{q^k \mid k \in \mathbb{Z}_{\geq 0}\}$, so that the category $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ is semisimple by Theorem 8.8 and Example 4 in Section 8 of [Kno07]. As before, we consider the sequence

$$b_n = \#\text{indecomposable summands in } 1^{\otimes n} \text{ counted with multiplicities.}$$

By Lemma 2.2 and the discussion in Subsection 3E.2, we have

$$b_n = \sum_{k \leq n} \Big( \#\mathcal{B}(k) \cdot \sum_{\chi} \chi(1) \Big),$$

where $\mathcal{B}(k) = \mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$ and $\chi$ runs over the irreducible characters of $\mathrm{GL}_k(\mathbb{F}_q)$. Therefore, in order to explicitly determine $b_n$, we need to

**(a)** compute the sum $cd(k) = \sum_\chi \chi(1)$ of the degrees of the irreducible characters of $\mathrm{GL}_k(\mathbb{F}_q)$;

**(b)** count the number of elements in $\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k) \cong \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k) \,/\, \mathrm{GL}_k(\mathbb{F}_q)$.

Since we assume that $q$ is odd, $cd(k)$ equals the number of symmetric matrices in $\mathrm{GL}_k(\mathbb{F}_q)$ by Theorem 3 in [Gow83], and so by Theorem 2 in [Mac69], we have

$$cd(k) = \#\text{symmetric matrices in } \mathrm{GL}_k(\mathbb{F}_q) = q^{\binom{k+1}{2}} \cdot \prod_{\substack{1 \leq i \leq k \\ i \text{ odd}}} (1 - \tfrac{1}{q^i}).$$

Next for $k \leq \ell \leq n$, let us write $\mathrm{Mat}^\ell(n \times \ell, \mathbb{F}_q)$ and $\mathrm{Mat}^k(k \times \ell, \mathbb{F}_q)$ for the sets of $n \times \ell$ matrices of rank $\ell$ and of $k \times \ell$ matrices of rank $k$, respectively, with entries in $\mathbb{F}_q$. Observe that for every pair of matrices

$M \in \mathrm{Mat}^k(k \times \ell, \mathbb{F}_q)$ and $N \in \mathrm{Mat}^\ell(n \times \ell, \mathbb{F}_q)$, we have $G(M) \circ G(N)^* \in \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$. Writing $\mathrm{Gr}^{\mathrm{IS}}_\ell(\mathbb{F}_q^n, \mathbb{F}_q^k)$ for the set of subspaces $V \in \mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$ with $\dim_{\mathbb{F}_q} V = \ell$, this gives rise to a bijection

$$\big(\mathrm{Mat}^k(k \times \ell, \mathbb{F}_q) \times \mathrm{Mat}^\ell(n \times \ell, \mathbb{F}_q)\big) \,/\, \mathrm{GL}_\ell(\mathbb{F}_q) \xrightarrow{1:1} \mathrm{Gr}^{\mathrm{IS}}_\ell(\mathbb{F}_q^n, \mathbb{F}_q^k),$$

where $\mathrm{GL}_\ell(\mathbb{F}_q)$ acts freely on $\mathrm{Mat}^k(k \times \ell, \mathbb{F}_q) \times \mathrm{Mat}^\ell(n \times \ell, \mathbb{F}_q)$ via $(g, M, N) \mapsto (Mg^{-1}, Ng^{-1})$. Note that this bijection is $\mathrm{GL}_k(\mathbb{F}_q)$-equivariant, where the action of $\mathrm{GL}_k(\mathbb{F}_q)$ on the left hand side is induced by the canonical action of $\mathrm{GL}_k(\mathbb{F}_q)$ on $\mathrm{Mat}^k(k \times \ell, \mathbb{F}_q)$ by left multiplication. If we fix sets of representatives $\mathrm{Mat}^\ell_0(n \times \ell, \mathbb{F}_q)$ for the $\mathrm{GL}_\ell(\mathbb{F}_q)$-orbits in $\mathrm{Mat}^\ell(n \times \ell, \mathbb{F}_q)$ and $\mathrm{Mat}^k_0(k \times \ell, \mathbb{F}_q)$ for the $\mathrm{GL}_k(\mathbb{F}_q)$-orbits in $\mathrm{Mat}^k(k \times \ell, \mathbb{F}_q)$ then it is straightforward to see that we obtain a bijection

$$\mathrm{Mat}^k_0(k \times \ell, \mathbb{F}_q) \times \mathrm{Mat}^\ell_0(n \times \ell, \mathbb{F}_q) \xrightarrow{1:1} \mathrm{Gr}^{\mathrm{IS}}_\ell(\mathbb{F}_q^n, \mathbb{F}_q^k) \,/\, \mathrm{GL}_k(\mathbb{F}_q),$$

which sends a pair $(M, N)$ to the $\mathrm{GL}_k(\mathbb{F}_q)$-orbit of $G(M) \circ G(N)^*$. (Later on, our preferred choice of orbit representatives $\mathrm{Mat}^\ell_0(n \times \ell, \mathbb{F}_q)$ will be the set of $n \times \ell$ matrices of rank $\ell$ in reduced column echelon form, and $\mathrm{Mat}^k_0(k \times \ell, \mathbb{F}_q)$ will be the set of $k \times \ell$ matrices of rank $k$ in reduced row echelon form.) Observe that $\mathrm{Mat}^\ell_0(n \times \ell, \mathbb{F}_q) \cong \mathrm{Mat}^\ell(n \times \ell, \mathbb{F}_q)/\mathrm{GL}_\ell(\mathbb{F}_q)$ is canonically in bijection with the set $\mathrm{Gr}_\ell(\mathbb{F}_q^n)$ of $\ell$-dimensional subspaces of $\mathbb{F}_q^n$ via $N \mapsto \mathrm{im}(N)$ (the image of $N$), and $\mathrm{Mat}^k_0(k \times \ell, \mathbb{F}_q) \cong \mathrm{Mat}^k(k \times \ell, \mathbb{F}_q)/\mathrm{GL}_k(\mathbb{F}_q)$ is in bijection with $\mathrm{Gr}_{\ell-k}(\mathbb{F}_q^\ell)$ via $M \mapsto \ker(M)$ (the kernel of $M$). The number of elements in $\mathrm{Gr}_b(\mathbb{F}_q^a)$ is given by the (Gaussian) $q$-binomial coefficient

$$\#\mathrm{Gr}_b(\mathbb{F}_q^a) = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_q = \frac{[a]_q!}{[b]_q![a-b]_q!} = \frac{[a]_q[a-1]_q \cdots [a-b+1]_q}{[b]_q[b-1]_q \cdots [1]_q},$$

where $[j]_q! = [j]_q[j-1]_q \cdots [1]_q$ and $[j]_q = \frac{q^j - 1}{q-1}$ for $j \in \mathbb{Z}_{\geq 1}$, and so we obtain

$$\#\mathrm{Gr}^{\mathrm{IS}}_\ell(\mathbb{F}_q^n, \mathbb{F}_q^k) \,/\, \mathrm{GL}_k(\mathbb{F}_q) = \left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right]_q \left[\begin{smallmatrix} \ell \\ \ell-k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right]_q \left[\begin{smallmatrix} \ell \\ k \end{smallmatrix}\right]_q, \qquad \#\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k) \,/\, \mathrm{GL}_k(\mathbb{F}_q) = \sum_{k \leq \ell \leq n} \left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right]_q \left[\begin{smallmatrix} \ell \\ k \end{smallmatrix}\right]_q.$$

In conclusion, we have

$$\begin{aligned} b_n &= \sum_{k \leq n} \#\mathrm{Gr}^{\mathrm{IS}}_0(\mathbb{F}_q^n, \mathbb{F}_q^k) \cdot cd(k) \\ &= \sum_{k \leq n} \Big( \sum_{k \leq \ell \leq n} \left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right]_q \left[\begin{smallmatrix} \ell \\ k \end{smallmatrix}\right]_q \Big) \cdot q^{\binom{k+1}{2}} \cdot \prod_{\substack{1 \leq i \leq k \\ i \text{ odd}}} (1 - \tfrac{1}{q^i}). \end{aligned}$$

This formula can be simplified as follows.

**Proposition 3E.6.** *For all $n \geq 0$, we have*

$$b_n = \sum_{k \leq n} \Big( \sum_{k \leq \ell \leq n} \left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right]_q \left[\begin{smallmatrix} \ell \\ k \end{smallmatrix}\right]_q \Big) \cdot q^{\binom{k+1}{2}} \cdot \prod_{\substack{1 \leq i \leq k \\ i \text{ odd}}} (1 - \tfrac{1}{q^i}) = \prod_{k=1}^{n} (q^k + 1),$$

*Proof.* First observe that for $0 \leq k \leq \ell \leq n$, we have

$$\left[\begin{smallmatrix} n \\ \ell \end{smallmatrix}\right]_q \left[\begin{smallmatrix} \ell \\ k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \left[\begin{smallmatrix} n-k \\ \ell-k \end{smallmatrix}\right]_q, \qquad q^{\binom{k+1}{2}} \cdot \prod_{\substack{1 \leq i \leq k \\ i \text{ odd}}} (1 - \tfrac{1}{q^i}) = \prod_{1 \leq i \leq k} q^i - \varepsilon_i =: p_k,$$

where $\varepsilon_i = 0$ if $i$ is even and $\varepsilon_i = 1$ if $i$ is odd, so we can rewrite $b_n$ as

$$b_n = \sum_{0 \leq k \leq n} \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \cdot \Big( \sum_{k \leq \ell \leq n} \left[\begin{smallmatrix} n-k \\ \ell-k \end{smallmatrix}\right]_q \Big) \cdot \prod_{1 \leq i \leq k} q^i - \varepsilon_i = \sum_{0 \leq k \leq n} p_k \cdot \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \cdot \Big( \sum_{0 \leq \ell \leq n-k} \left[\begin{smallmatrix} n-k \\ \ell \end{smallmatrix}\right]_q \Big).$$

We prove the claim of the proposition by induction on $n$, using the $q$-Pascal identity

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q = \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]_q + q^{n-k} \cdot \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]_q$$

and starting from the observation that $b_0 = 1$. (We have $\left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]_q = 1$ and $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]_q = 0$ if $b < 0$ or $b > a$ by convention.) For $n > 0$, we compute

$$\begin{aligned} b_n &= \sum_{0 \leq k \leq n} p_k \cdot \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \cdot \Big( \sum_{0 \leq \ell \leq n-k} \left[\begin{smallmatrix} n-k \\ \ell \end{smallmatrix}\right]_q \Big) \\ &= \Big( \sum_{0 \leq k \leq n-1} p_k \cdot \Big( \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]_q + q^{n-k} \cdot \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix}\right]_q \Big) \cdot \Big( \sum_{0 \leq \ell \leq n-k} \left[\begin{smallmatrix} n-k \\ \ell \end{smallmatrix}\right]_q \Big) \Big) + p_n \\ &= \Big( \sum_{0 \leq k \leq n-1} p_k \cdot \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix}\right]_q \cdot \Big( 1 + \sum_{0 \leq \ell \leq n-k-1} \left[\begin{smallmatrix} n-k-1 \\ \ell \end{smallmatrix}\right]_q + q^{n-k-\ell} \cdot \left[\begin{smallmatrix} n-k-1 \\ \ell-1 \end{smallmatrix}\right]_q \Big) \Big) \end{aligned}$$

$$
\begin{aligned}
&\quad + \Big( \sum_{0\le k\le n-1} q^{n-k}\cdot p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k-1}_q \cdot \Big( \sum_{0\le \ell\le n-k} \genfrac{[}{]}{0pt}{}{n-k}{\ell}_q \Big)\Big) + p_n \\
&= \Big( \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \Big) + \Big( \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \Big( \sum_{0\le \ell\le n-k-1} \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q \Big)\Big) \\
&\quad + \Big( \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \Big( \sum_{0\le \ell\le n-k-1} q^{n-k-\ell}\cdot \genfrac{[}{]}{0pt}{}{n-k-1}{\ell-1}_q \Big)\Big) \\
&\quad + \Big( \sum_{0\le k\le n-1} q^{n-k}\cdot p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k-1}_q \cdot \Big( \sum_{0\le \ell\le n-k} \genfrac{[}{]}{0pt}{}{n-k}{\ell}_q \Big)\Big) + p_n \\
&= b_{n-1} + c_{n-1} + d_{n-1} + p_n,
\end{aligned}
$$

where we set

$$
\begin{aligned}
c_{n-1} &= \Big( \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} q^{n-k-\ell}\cdot \genfrac{[}{]}{0pt}{}{n-k-1}{\ell-1}_q \Big) + \Big( \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \Big), \\
d_{n-1} &= \sum_{0\le k\le n-1} q^{n-k}\cdot p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k-1}_q \cdot \sum_{0\le \ell\le n-k} \genfrac{[}{]}{0pt}{}{n-k}{\ell}_q .
\end{aligned}
$$

Using the fact that $p_{k+1} = (q^{k+1} - \varepsilon_{k+1}) \cdot p_k$, we obtain

$$
\begin{aligned}
d_{n-1} &= \sum_{0\le k\le n-1} q^{n-k} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k-1}_q \cdot \sum_{0\le \ell\le n-k} \genfrac{[}{]}{0pt}{}{n-k}{\ell}_q \\
&= \sum_{0\le k\le n-2} q^{n-k-1} p_{k+1} \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q \\
&= \sum_{0\le k\le n-2} q^{n-k-1} \cdot (q^{k+1} - \varepsilon_k) \cdot p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q \\
&= q^n \cdot \sum_{0\le k\le n-2} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q \\
&\qquad - \sum_{0\le k\le n-2} q^{n-k-1} \varepsilon_k p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q \\
&= q^n b_{n-1} - q^n p_{n-1} - \Big( \sum_{0\le k\le n-1} q^{n-k-1} \varepsilon_k p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q \Big) + \varepsilon_n p_{n-1} \\
&= q^n b_{n-1} - p_n - \Big( \sum_{0\le k\le n-1} q^{n-k-1} \varepsilon_k p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q \Big),
\end{aligned}
$$

and we further compute

$$
\begin{aligned}
c_{n-1} &= \Big( \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} q^{n-k-\ell}\cdot \genfrac{[}{]}{0pt}{}{n-k-1}{\ell-1}_q \Big) + \Big( \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \Big) \\
&= \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k} q^{n-k-\ell}\cdot \genfrac{[}{]}{0pt}{}{n-k-1}{\ell-1}_q \\
&= \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} q^{n-k-\ell-1}\cdot \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q .
\end{aligned}
$$

Thus, if we define

$$
z_{n-1} \coloneqq c_{n-1} + d_{n-1} - q^n b_{n-1} + p_n = \sum_{0\le k\le n-1} p_k \cdot \genfrac{[}{]}{0pt}{}{n-1}{k}_q \cdot \sum_{0\le \ell\le n-k-1} \big( q^{n-k-\ell-1} - q^{n-k-1}\varepsilon_{k+1} \big) \cdot \genfrac{[}{]}{0pt}{}{n-k-1}{\ell}_q
$$

then it follows that

$$
b_n = b_{n-1} + c_{n-1} + d_{n-1} + p_n = (q^n + 1)\cdot b_{n-1} + z_{n-1},
$$

and so it remains to show that $z_n = 0$ for all $n \ge 0$. For $0 \le k \le n$, let

$$
\begin{aligned}
s_{k,n} &\coloneqq p_k \cdot \genfrac{[}{]}{0pt}{}{n}{k}_q \cdot \sum_{0\le \ell\le n-k} \big( q^{n-k-\ell} - q^{n-k}\varepsilon_{k+1} \big) \cdot \genfrac{[}{]}{0pt}{}{n-k}{\ell}_q \\
&= p_k \cdot \sum_{0\le \ell\le n-k} \big( q^{\ell} - q^{n-k}\varepsilon_{k+1} \big) \cdot \genfrac{[}{]}{0pt}{}{n-k}{\ell}_q
\end{aligned}
$$

so that $z_n = \sum_{0\le k\le n} s_{k,n}$, and observe that if $n$ is even then $s_{n,n} = 0$. If $k < n$ is even then we have

$$\begin{aligned} s_{k,n} + s_{k+1,n} &= \left( p_k \cdot \left[\begin{smallmatrix} n \\ k \end{smallmatrix}\right]_q \cdot \sum_{0\le \ell\le n-k} \left(q^\ell - q^{n-k}\right) \cdot \left[\begin{smallmatrix} n-k \\ \ell \end{smallmatrix}\right]_q \right) \\ &\quad + \left( p_{k+1} \cdot \left[\begin{smallmatrix} n \\ k+1 \end{smallmatrix}\right]_q \cdot \sum_{0\le \ell\le n-k-1} q^\ell \cdot \left[\begin{smallmatrix} n-k-1 \\ \ell \end{smallmatrix}\right]_q \right) \\ &= \left( p_k \cdot \sum_{0\le \ell\le n-k} q^\ell \cdot \left(1 - q^{n-k-\ell}\right) \cdot \tfrac{[n]_q!}{[k]_q![\ell]_q![n-k-\ell]_q!} \right) \\ &\quad + \left( p_k \cdot \sum_{0\le \ell\le n-k-1} q^\ell \cdot \left(q^{k+1} - 1\right) \cdot \tfrac{[n]_q!}{[k+1]_q![\ell]_q![n-k-\ell-1]_q!} \right) \\ &= p_k \cdot q^{n-k} \cdot (1 - q^0) \cdot \tfrac{[n]_q!}{[k]_q![\ell]_q!} = 0. \end{aligned}$$

We conclude that $z_n = \sum_{0\le k\le n} s_{k,n} = 0$ for all $n \ge 0$, hence $b_n = (q^n + 1) \cdot b_{n-1}$ for all $n > 0$, and by induction, it follows that $b_n = \prod_{k=1}^{n}(q^k + 1)$, as claimed. □

If we set

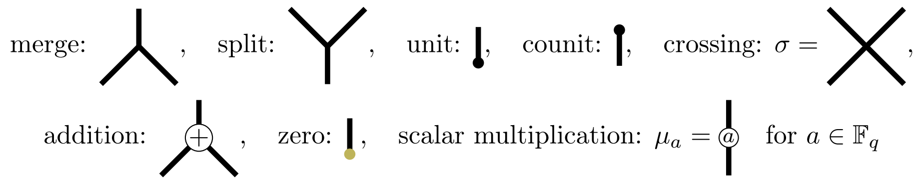
$$c = \lim_{n\to\infty} \prod_{k=1}^{n} (1 + \tfrac{1}{q^k}) = \tfrac{1}{2}\mathrm{QPochhammer}(-1, 1/q)_\infty$$

(the q-Pochhammer symbol) then Proposition 3E.6 implies that

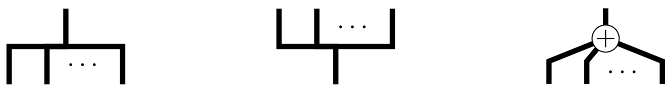
$$b_n \sim c \cdot q^{\frac{n(n+1)}{2}}, \qquad \sqrt[n]{b_n} \sim q^{\frac{n+1}{2}},$$

and this establishes the claim in (1.1).

3E.4. *Diagrammatic interpretation.* The category $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ admits a diagrammatic description, which we recall below, following Subsection 5.2 in [**EAH22**]. Namely, $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ is generated by the object $1 = \bullet$ and by the following homomorphisms.



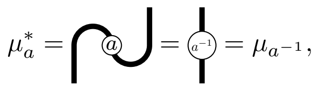

The relations are defined precisely so as to make $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ the universal rigid symmetric monoidal category generated by an $\mathbb{F}_q$-linear Frobenius space (i.e. a Frobenius algebra object with a compatible $\mathbb{F}_q$-module structure); c.f. Definition 5.1.1 in [**EAH22**]. In particular, the merge and addition homomorphisms satisfy the usual associativity relations and the split homomorphism satisfies coassociativity, so that we can define iterated merge, split and addition homomorphisms. These will be denoted by the following diagrams:



The evaluation and coevaluation for the generating object $1 = \bullet$ are given by


and we introduce the additional notations


Using the relations (DLin$_3$) and (DRel$_1$) in [**EAH22**], it is straightforward to see that for $a \in \mathbb{F}_q^\times$, the dual of $\mu_a$ is given by

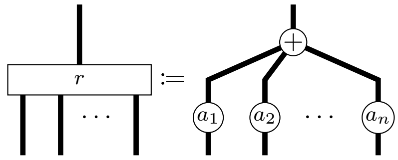



and again using (DLin$_3$), we have

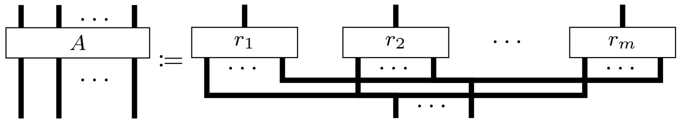

Thus, the dual of a homomorphism in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$, represented by a diagram, can be computed by

- replacing all occurences of $\mu_0$ by ,
- replacing all occurences of $\mu_a$ by $\mu_{a^{-1}}$ for $a \in \mathbb{F}_q^\times$,
- turning the diagram upside down.

For every row vector $r = (a_1, \ldots, a_n)$ with entries in $\mathbb{F}_q$, the homomorphism $G(r)\colon n \to 1$ defined in Remark 3E.2 corresponds to the diagram

(3E.7)


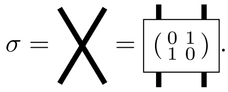

Furthermore, for an $m \times n$ matrix $A$ with rows $r_1, \ldots, r_m$, the homomorphism $G(A)\colon n \to m$ is represented by the diagram

(3E.8)


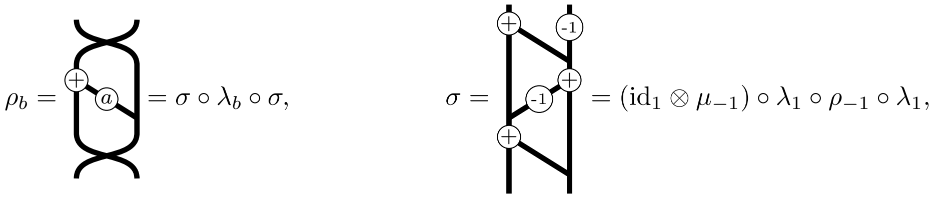

With these notations in place, we can now translate the sandwich cellular structure defined in Subsection 3E.2 into diagrammatics. Recall that we write $\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$ for the set of injective-surjective subspaces of $\mathbb{F}_q^n \oplus \mathbb{F}_q^k$ and $\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$ for the set of $\mathrm{GL}_k(\mathbb{F}_q)$ orbits in $\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^n, \mathbb{F}_q^k)$, and similarly for surjective-injective subspaces. Further recall that every subspace $V \in \mathrm{Gr}(\mathbb{F}_q^m, \mathbb{F}_q^n)$ can be factored uniquely as $V = V_2 \circ G(M) \circ V_1$, with $V_1 \in \mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$, $V_2 \in \mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$ and $M \in \mathrm{GL}_k(\mathbb{F}_q)$, for some $k \leq \min\{m, n\}$. Therefore, in order define bases for Hom-spaces in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ consisting of diagrams in a certain "standard form", it suffices to diagrammatically describe the groups $\mathrm{GL}_k(\mathbb{F}_q)$ and the top and bottom homomorphisms corresponding to $\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q k)$ and $\mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$, respectively.

The diagrammatic description of the the groups $\mathrm{GL}_k(\mathbb{F}_q)$ is straightforward, using the observation that $\mathrm{GL}_k(\mathbb{F}_q)$ is generated by elementary matrices (i.e. elementary row or column operations). Accordingly, we call a $\mathrm{GL}_*(\mathbb{F}_q)$***-diagram*** any diagram in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ that is locally generated by the scalar multiplication diagrams $\mu_a$, for $a \in \mathbb{F}_q^\times$, along with the elementary matrix diagrams


for $a \in \mathbb{F}_q$. Alternatively, a $\mathrm{GL}_*(\mathbb{F}_q)$-diagram is locally generated by the scalar multiplication diagrams $\mu_a$ (for $a \in \mathbb{F}_q^\times$), along with the elementary matrix diagrams $\lambda_b$ (for $b \in \mathbb{F}_q$) and the crossing




Indeed, the equivalence of the generating sets is a consequence of the relations




which in turn follow from the matrix equations

$$\left(\begin{smallmatrix} 1 & 0 \\ b & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \qquad \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),$$

cf. Proposition 6.2.10 in [**EAH22**]. A more detailed standard form for $\mathrm{GL}_*(\mathbb{F}_q)$-diagrams can be obtained using the Bruhat decomposition for finite general linear groups, but we will not discuss the details here.

In order to diagrammatically describe the bottom (and top) homomorphisms corresponding to $\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$ (and $\mathrm{Gr}_0^{\mathrm{SI}}(\mathbb{F}_q^k, \mathbb{F}_q^n)$), let us write

$$\mathrm{Mat}_0^\ell(m \times \ell, \mathbb{F}_q) = \big\{ m \times \ell \text{ matrices of rank } \ell \text{ in reduced column echelon form} \big\},$$
$$\mathrm{Mat}_0^k(k \times \ell, \mathbb{F}_q) = \big\{ k \times \ell \text{ matrices of rank } k \text{ in reduced row echelon form} \big\},$$

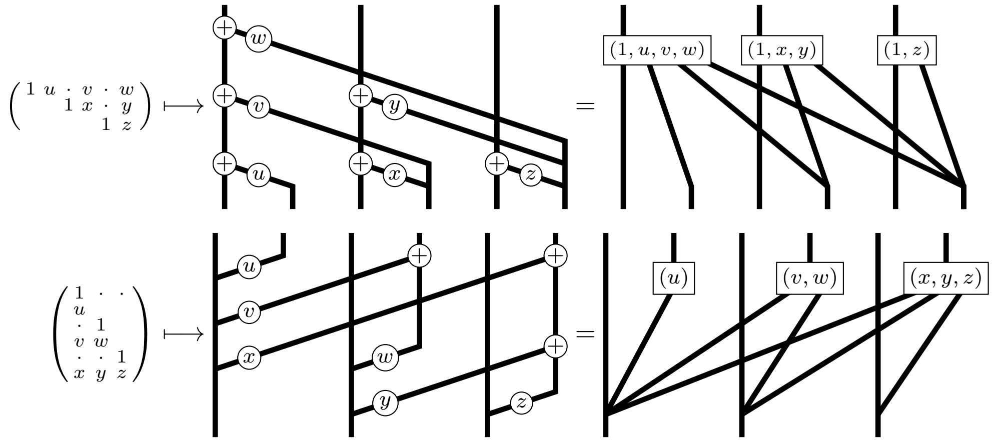
for $k \leq \ell \leq m$. Recall from Subsection 3E.2 that there is a bijection

$$\mathrm{Mat}_0^k(k \times \ell, \mathbb{F}_q) \times \mathrm{Mat}_0^\ell(m \times \ell, \mathbb{F}_q) \xrightarrow{1:1} \mathrm{Gr}_\ell^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k) \,/\, \mathrm{GL}_k(\mathbb{F}_q),$$

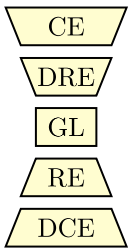
which sends a pair $(M, N)$ to the $\mathrm{GL}_k(\mathbb{F}_q)$-orbit of $G(M) \circ G(N)^*$, so we can define a set of representatives $\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$ for the $\mathrm{GL}_k(\mathbb{F}_q)$-orbits in $\mathrm{Gr}^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$ via

$$\mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k) = \bigsqcup_{k \leq \ell \leq n} \big\{ G(M) \circ G(N)^* \;\big|\; M \in \mathrm{Mat}_0^k(k \times \ell, \mathbb{F}_q), N \in \mathrm{Mat}_0^\ell(m \times \ell, \mathbb{F}_q) \big\}.$$

Then, in order to diagrammatically describe the homomorphism in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ corresponding to an injective-surjective subspace $V = G(M) \circ G(N)^* \in \mathrm{Gr}_0^{\mathrm{IS}}(\mathbb{F}_q^m, \mathbb{F}_q^k)$, it suffices to diagrammatically describe the homomorphisms $G(M)$ and $G(N)$ corresponding to matrices of full rank in reduced row echelon form or column echelon form, respectively. (As explained above, the diagram corresponding to $G(N)^*$ can be obtained essentially by turning the diagram corresponding to $G(N)$ upside down.) The diagrams in question were defined in (3E.8), but in the special case of matrices in reduced row echelon form or column echelon form, they can be further simplified, as shown in the following examples (using the notation introduced in (3E.7)).




Any diagram of this form will be called a ***row echelon diagram*** or ***column echelon diagram***, respectively, and their duals will be called dual row echelon diagrams or dual column echelon diagrams. With these conventions, the standard form for diagrammatic homomorphisms in $\mathrm{Rep}(\mathrm{GL}_t(\mathbb{F}_q))$ is as follows:




column echelon diagram,
dual row echelon diagram,
$\mathrm{GL}_*(\mathbb{F}_q)$-diagram,
row echelon diagram
dual column echelon diagram

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