All knots are equal (for my favorite invariants)

Or: Jones & friends: terrible at detection, wonderful anyway

~~Accept~~Change what you cannot ~~change~~accept

I report on work of Kelomäki–Lacabanne–Vaz–Zhang & many others, e.g. cake by Clancy–Sisto

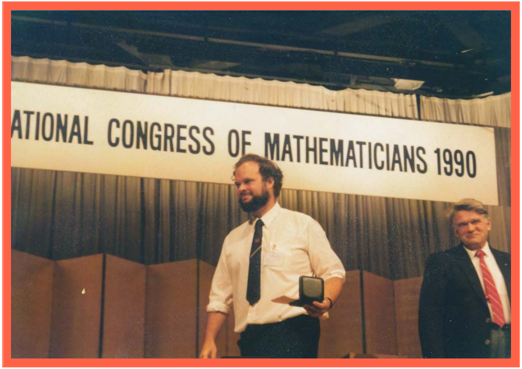
Quantum invariants

- The icing is in the shape of a trefoil knot
- The cake itself has the shape of a trefoil knot complement
- What else? It contains $\approx$ 500 grams of butter, 500 grams of sugar and 7 eggs
- From https://alexsisto.wordpress.com/2012/05/02

Quantum invariants

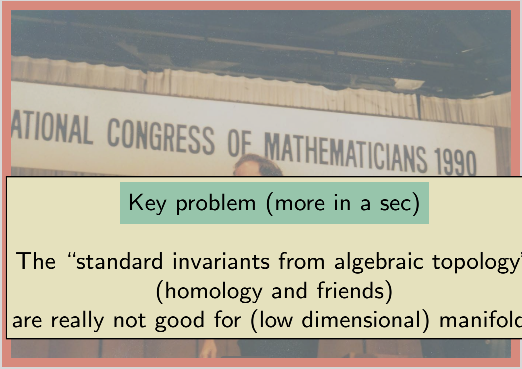
- The knot complement is a complete knot invariant
- Great!? Well, the decision problem is not trackable
- Problem "This is not computable"

Quantum invariants



- Kyoto 1990 Jones receives the fields medal (with Faddeev in the background)
- This talk What about computable knot invariants?
- No cake! Too bad...

Quantum invariants



Key problem (more in a sec)

The "standard invariants from algebraic topology"
(homology and friends)
are really not good for (low dimensional) manifolds

- Kyoto 1990 Jones receives the fields medal (with Faddeev in the background)
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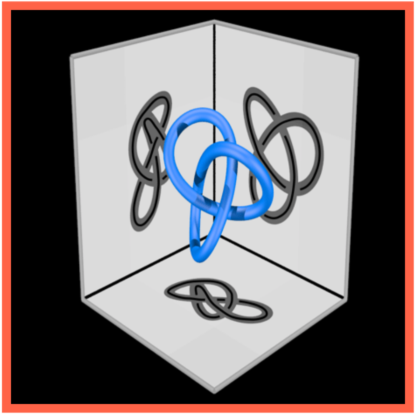
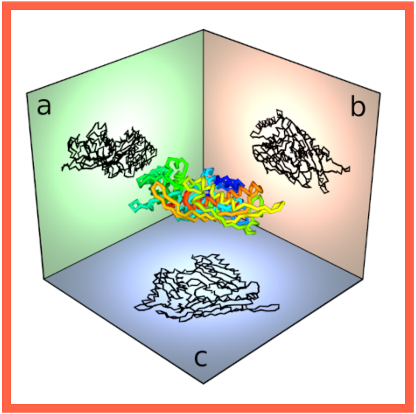
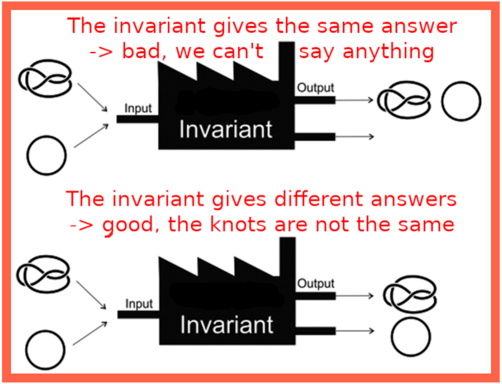
Quantum invariants





- Knot = closed string (a circle S^1) in three spaces; link = multiple components
- Knots are studied by projections to the plane Shadows
- Knots/links are the basic building blocks of low dimensional manifolds

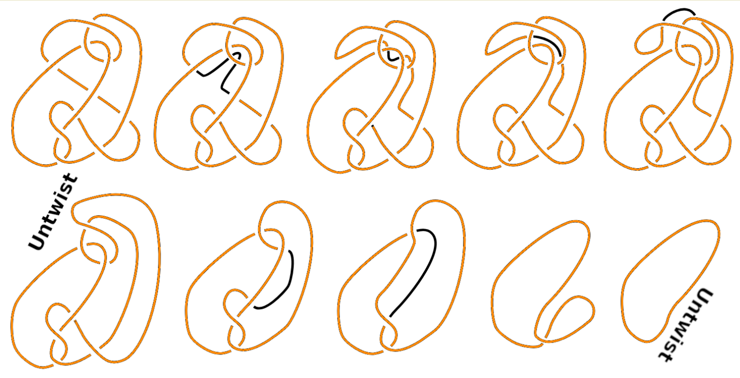
Quantum invariants



- In math knot theory started in the early 20th century
- Topologists from ~1900-1980 studied knots from the point of view of invariants from homology theory
- Problem The invariants obtained are not particularly strong

Quantu

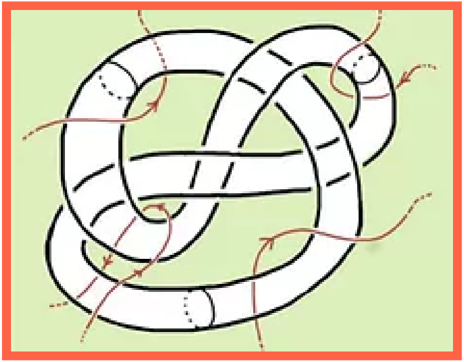
Even the unknotting problem is tricky



In general, knot theory was in need of new invariants
since the “standard invariants from algebraic topology”
(homology and friends)
are really not good for knots

- In
- To
  in
- Problem The invariants obtained are not particularly strong

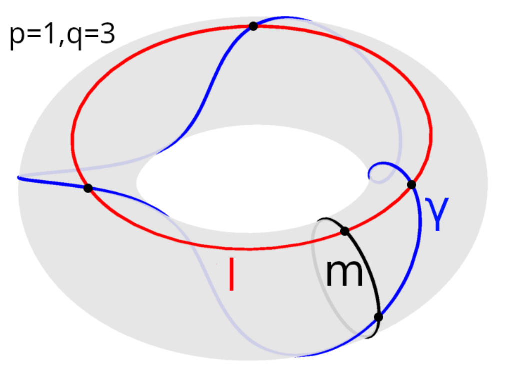
Quantum invariants



- A knot complement $S^3 \setminus \text{int}(K)$ is a 3mfd bounding a torus
- Idea Glue back in a solid torus ST, but "twisted"
- Any such gluing is determined by the image of the meridian m, and m goes to some simple closed curve γ in $T = \partial ST$, and it hence suffices to describe γ

Quantum invariants

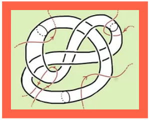
Write $[\gamma] = p \cdot [l] + q \cdot [m] \in H_1(\partial T)$



Surgery: We take out a torus T, fix γ determined by p, q and glue the meridian m of T back in on γ

- A knot c
- Idea Glu
- Any such gluing is determined by the image of the meridian m, and m goes to some simple closed curve γ in $T = \partial ST$, and it hence suffices to describe γ

Quantum invariants



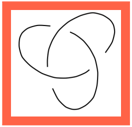
Every closed, orientable, connected 3mfd can be obtained by Dehn surgery, that is:

(i) Pick a finite collection of knots in S^3

(ii) Pick a surgery coefficient (p, q) for each knot

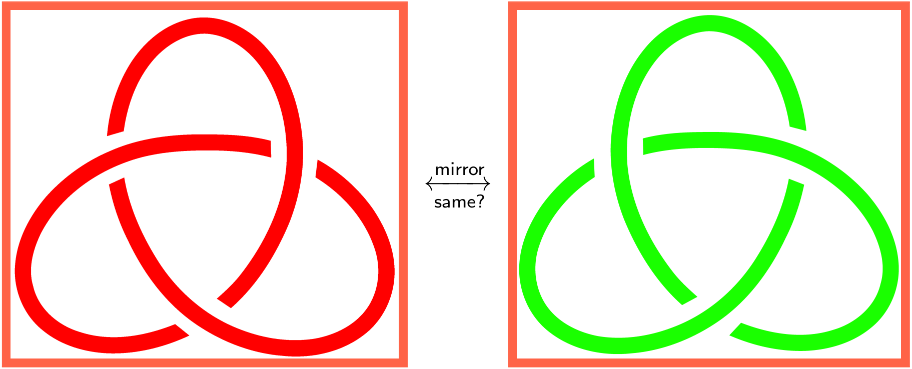
(iii) Perform the "remove-insert" surgery

- Every surgery on a knot gluing meridian to longitude gives a homology sphere



- Homology is "really bad"!

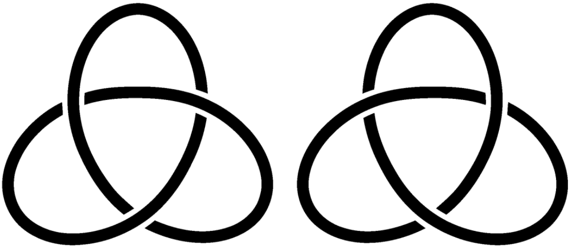
Quantum invariants



- Problem Deciding whether two knot projections are the same knot is difficult
- Task Find an invariant. Sounds easy? Well, most knot invariants are pretty bad...so: find a ‘good’ knot invariant
- Example There was no knot invariant that can distinguish the above knots

Jones' revolution (quantum invariants)

Left = right-handed trefoil? No!



- The left-handed trefoil has Jones polynomial $-q^4 + q^3 + q$
- The right-handed trefoil has Jones polynomial $-q^{-4} + q^{-3} + q^{-1}$
- Thus, they are different

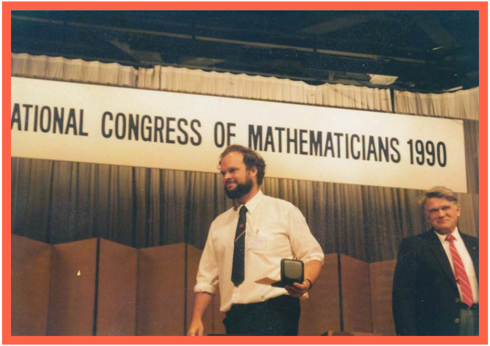
A zoo of quantum invariants For any semisimple Lie algebra and any representation:

Jones ∼1985 + friends There are polynomial knot/3mfd invariants

Khovanov ∼1999 + friends There are homological knot/3mfd/4mfd invariants

- Example There was no knot invariant that can distinguish the above knots

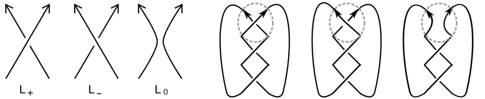
Quantum invariants



- Kyoto 1990 Jones receives the fields medal (with Faddeev in the background)
- Quote "Jones discovered an astonishing relationship between von Neumann algebras and geometric topology. As a result, they found a new polynomial invariant for knots and links in 3-space."
- Today The focus is on the quantum knot invariants à la Jones

Quantum invariants

Example (of quantum invariants)



- Alexander $(q^{1/2} - q^{-1/2}) \cdot \Delta_{L_0}(q) = \Delta_{L_+}(q) - \Delta_{L_-}(q)$
- Jones polynomial:
  - Skein relation $(q^{1/2} - q^{-1/2}) \cdot J_{L_0}(q) = q^{-1} \cdot J_{L_+}(q) - q \cdot J_{L_-}(q)$
  - Hecke algebra of the braid group
  - Quantum field theory as the unknot normalized vacuum expectation value of the Wilson loop operator in $SU(2)$ Chern–Simons gauge theory
- HOMFLY-PT: $z \cdot H_{L_0}(q) = a \cdot H_{L_+}(q) - a^{-1} \cdot H_{L_-}(q)$
- Khovanov homology – categorification of the Jones polynomial

Everyone loves them (I have spend **1/4 of a century** studying them) and they triggered a lot of research in

low dim topology, mathematical physics, modular Lie theory, ...

Question How good are these invariants (say, on prime knots)?

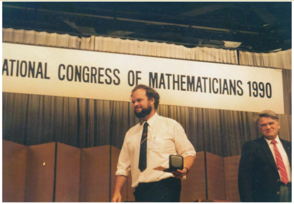
▶ Ky... ...kground)

▶ Qu... ...eumann alge... ...nomial inva...

▶ Today The focus is on the quantum knot invariants à la Jones

Quantum invariants

They are loved because they relate many fields



- Kyoto 1990 Jones receives the fields medal (with Faddeev in the background)
- Quote “Jones discovered an astonishing relationship between von Neumann algebras and geometric topology. As a result, they found a new polynomial invariant for knots and links in 3-space.”

But how do they actually perform?

- Kyoto 199 … the background)
- Quote “J … von Neumann algebras and geometric topology. As a result, they found a new polynomial invariant for knots and links in 3-space.”
- Today The focus is on the quantum knot invariants à la Jones

Big data and knots

$R =$ , $R^{-1} =$ , $cap = \frown$, $cup = \smile$, $id = |$.

We associate these to linear maps (matrices upon choice of basis) denoted with the same symbols

(2D.1) $R, R^{-1}: V_q \otimes V_q \to V_q \otimes V_q$, $cap: V_q \otimes V_q \to \mathbb{C}(q)$, $cup: \mathbb{C}(q) \to V_q \otimes V_q$, $id: V_q \to V_q, v \mapsto v$.

$\leftrightsquigarrow$

$0 \otimes cap \otimes 0$
$\circ$
$1 \otimes cap \otimes 1$
$\circ$
$2 \otimes cap \otimes 2$
$\circ$
$1 \otimes R^{-1} \otimes 3$
$\circ$
$0 \otimes R \otimes 4$
$\circ$
$1 \otimes R^{-1} \otimes 3$
$\circ$
$0 \otimes R \otimes 4$
$\circ$
$2 \otimes cup \otimes 2$
$\circ$
$1 \otimes cup \otimes 1$
$\circ$
$0 \otimes cup \otimes 0$

- Construction of quantum invariants $(\mathfrak{g}, V_q)$ See above; here V_q is a representation of some semisimple Lie algebra $\mathfrak{g}$
- Black box Quantum groups give us the matrices
- Categorification There are also homology versions (defined similarly)

Example

For the Jones polynomial J take $\mathfrak{g} = \mathfrak{sl}_2$, and $V_q = \mathbb{C}^2$

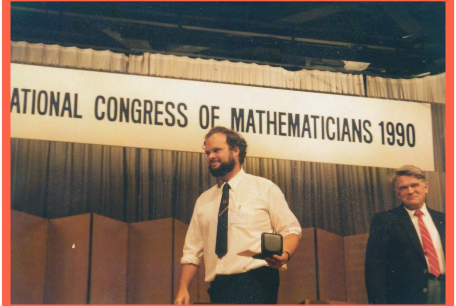
The R matrix is

$$R = \begin{pmatrix} q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & q & 0 \\ 0 & q & q^{1/2} - q^{3/2} & 0 \\ 0 & 0 & 0 & q^{1/2} \end{pmatrix}$$

$q = 1$ gives the swap map

- Categorification There are also homology versions (defined similarly)

Big data and knots



(a) We start with the ***Jones polynomial*** or ***A1 invariant*** $(\mathfrak{sl}_2, \mathbb{C}^2, 0)$ (for the vector representation). This is our reference invariant.

(b) We investigate the ***2-colored Jones polynomial*** or ***B1 invariant*** $(\mathfrak{sl}_2, \mathrm{Sym}^2\mathbb{C}^2, 0)$ (for the simple three-dimensional representation). This is coloring.

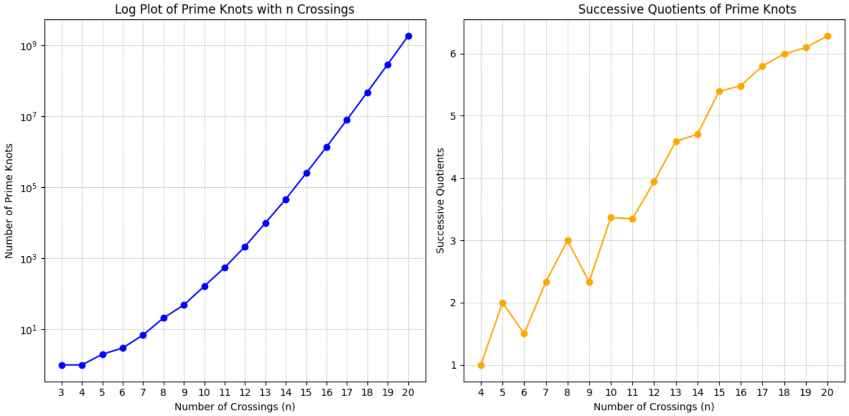
(c) We look at the ***A2 invariant*** $(\mathfrak{sl}_3, \mathbb{C}^3, 0)$ (for the vector representation). This is a rank increase.

(d) We then look at ***Khovanov homology*** or ***A1^c invariant*** $(\mathfrak{sl}_2, \mathbb{C}^2, 1)$ (for the vector representation). This is categorification.

(e) Finally, we have the most classical knot polynomial, the ***Alexander polynomial*** or ***isotropic A1 invariant*** $(\mathfrak{gl}_{1|1}, \mathbb{C}^{1|1}, 0)$ (for the vector representation). Here we leave the realm of Lie algebras.

Bi

Crucial



For this to work we need a lot of data; and we are lucky:

Ernst–Sumners ∼1987 The number of knots grows exponential

(a

(b

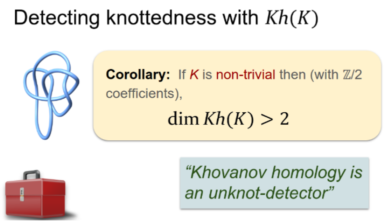
(c

(d

(e) Finally, we have the most classical knot polynomial, the ***Alexander polynomial*** or ***isotropic A1 invariant*** $(\mathfrak{gl}_{1|1}, \mathbb{C}^{1|1}, 0)$ (for the vector representation). Here we leave the realm of Lie algebras.

Big data and knots

Kronheimer–Mrowka gave a beautiful ICM talk about this (and related) breakthrough(s) Google 'Kronheimer Mrowka ICM 2018'



- First measure Put all (prime) knots in a bag, grab one randomly, how likely distinguishes, say, J the knot (from all others)?
- More formally What is

$$\lim_{n\to\infty} \#(\text{different } J \text{ with } \leq n \text{ crossings})/\#(\text{knots with } \leq n \text{ crossings})?$$

Big data and knots

Kro…

brea…

Small number coincidences?

KHOVANOV HOMOLOGY DETECTS:

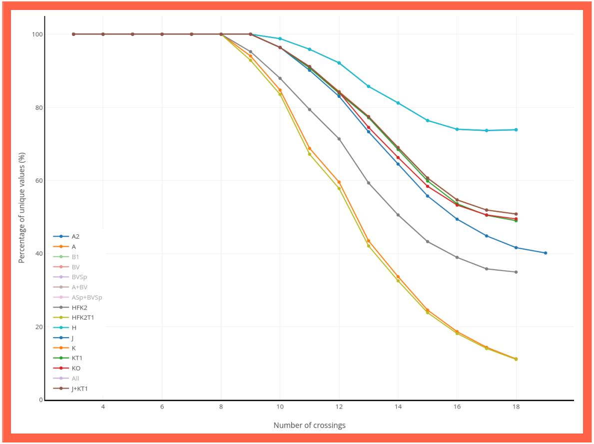
- The unknot: Kronheimer–Mrowka (2010)
- The unlink Hedden–Ni (2013), Batson–Seed (2015)
- The trefoils Baldwin–Sivek (2018)
- The Hopf link Baldwin–Sivek–Xie (2018)
- $2_1\#2_1$, the torus link $T(2,4)$ Xie–Zhang (2019)
- Split links Lipshitz–Sarkar (2019)
- The torus link $T(2,6)$ Martin (2020)
- $L6n1$ Xie–Zhang (2020)
- $L7n1$, $2_1\#3_1$ Li–Xie–Zhang (2020)
- Cinquefoil $T(5,2)$, non-fibered Baldwin, Siwek (2022)

▶ First meas… …omly, how likely distinguishe…

▶ More formally What is

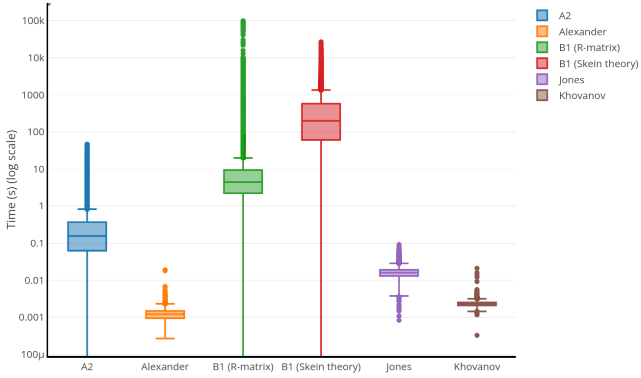
$$\lim_{n\to\infty} \#(\text{different } J \text{ with} \leq n \text{ crossings})/\#(\text{knots with} \leq n \text{ crossings})?$$

Big data and knots



- 1/4 century wasted!? They all distinguish knots with probability zero
- Data visualization gives us this conjecture and we can prove it for some of them

Even worse They all drop exponentially fast (proven in some cases)



If that is true, then the additional measure
we would use is the computational complexity (in the number of crossings)

Invariant knot	A	A1	B1	J	K
Capital O	polynomial	$\approx 3^{\sqrt{n}}$	$\approx 3^{\sqrt{n}}$	$\approx 2^{\sqrt{n}}$	$\approx 2^{n}$ (maybe better)

Alexander is then by far the best

Big data and

One can even prove that!

Theorem 1.1. *All of the marked invariants in Table 1 detect alternating links with probability zero. Even stronger, the detection probability decays exponentially with the crossing number.*

Polynomial	P(detection) = 0?	Homology	P(detection) = 0?
Jones	Yes	Khovanov over $\mathbb{Z}$	Yes
Alexander	Yes	Odd Khovanov over $\mathbb{Z}$	Yes
SL_N	Yes	HFK over $\mathbb{F}_2$	Yes
HOMFLYPT	Yes	Khovanov–Rozansky over $\mathbb{Z}$	Likely
Jones all colors	Yes	HOMFLYPT over $\mathbb{Z}$ (for knots)	Likely
SL_N for (1^k)	Yes	(1^k) or (k) versions of these	Likely
SL_N for (k)	Yes	Khovanov–Floer theories over $\mathbb{F}_2$	Likely
SL_N for $(N-1,1)$	Yes	Other	P(detection) = 0?
HOMFLYPT for (1^k)	Yes	Signature	Yes
HOMFLYPT for (k)	Yes	Determinant	Yes
Kauffman	Yes	Double branched cover	Yes
SO_{2N+1}	Yes	HF of double branched cover	Yes
SP_{2N}	Yes	Algebraic concordance	Yes
SO_{2N}	Yes	Finite type invariants of degree ≤ 10	Yes
G_2	Yes	Many invariants that we forgot to add	Depends, but likely

TABLE 1. Summary of our main results. Green entries ("Yes") are proved in this paper. Blue entries ("Likely") indicate invariants for which we outline proof strategies and supply big data evidence, though final arguments remain open.

W. T. Tutte

21 languages

Article Talk Read Edit View history Tools

From Wikipedia, the free encyclopedia

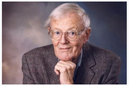
William Thomas Tutte OC FRS FRSC (/tʌt/; 14 May 1917 – 2 May 2002) was an English and Canadian code breaker and mathematician. During the Second World War, he made a fundamental advance in cryptanalysis of the Lorenz cipher, a major Nazi German cipher system which was used for top-secret communications within the Wehrmacht High Command.

The high-level, strategic nature of the intelligence obtained from Tutte's crucial breakthrough, in the bulk decrypting of Lorenz-enciphered messages specifically, contributed greatly, and perhaps even decisively, to the defeat of Nazi Germany.[2][3][4] He also had a number of significant mathematical accomplishments, including foundation work in the fields of graph theory and matroid theory.[5][6]

Tutte's research in the field of graph theory proved to be of remarkable importance. At a time when graph theory was still a primitive subject, Tutte

W. T. Tutte

OC FRS FRSC



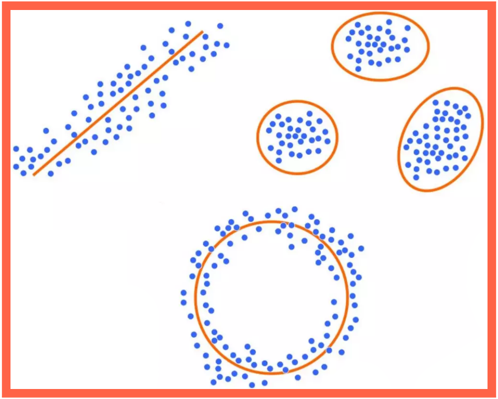
Born	William Thomas Tutte 14 May 1917 Newmarket, Suffolk, England
Died	2 May 2002 (aged 84)

How? Think about it for 10+ years, ask ChatGPT, it finds it in the literature (Tutte ≈ 1962)

done (maybe...)

- 1/4 cent... ity zero
- Data visualization gives us this conjecture and we can prove it for some of them

Big data and knots - TDA



- TDA (topological data analysis) is the art of finding the shape of data
- Question What shape are quantum knot invariants?
- Question Can the shape measure how good they are?

Big data and knots - TDA

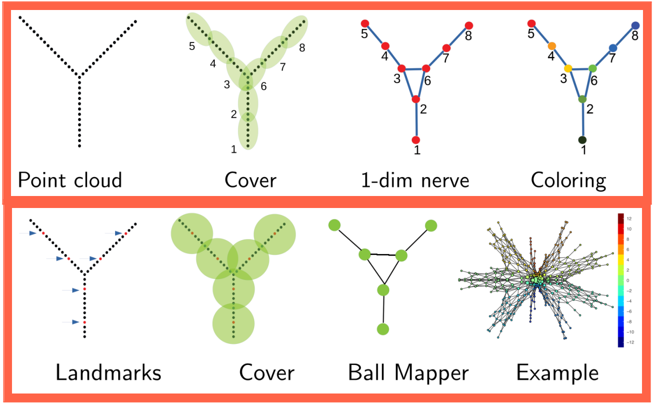
Knots form point clouds!

	q^{-3}	q^{-2}	q^{-1}	q^{0}	q^{1}	q^{2}	q^{3}	q^{4}	q^{5}	q^{6}	q^{7}
$J(0_1)$	0	0	0	1	0	0	0	0	0	0	0
$J(\mathrm{mir}(3_1))$	0	0	0	0	1	0	1	-1	0	0	0
$J(4_1)$	0	1	-1	1	-1	1	0	0	0	0	0
$J(\mathrm{mir}(5_1))$	0	0	0	0	0	1	0	1	-1	1	-1
$J(\mathrm{mir}(5_2))$	0	0	0	0	1	-1	2	-1	1	-1	0
$J(\mathrm{mir}(6_1))$	0	1	-1	2	-2	1	-1	1	0	0	0
$J(\mathrm{mir}(6_2))$	0	0	1	-1	2	-2	2	-2	1	0	0
$J(6_3)$	-1	2	-2	3	-2	2	-1	0	0	0	0

These are vectors in a 11d space

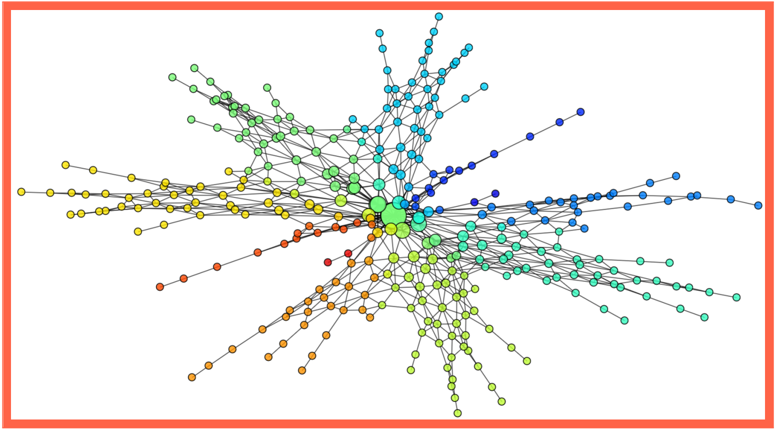
- Question Can the shape measure how good they are?

Big data and knots - TDA



- (Ball) Mapper = a way to turn point clouds into a graph
- Coloring gives additional information
- We see this in examples momentarily

Big data and knots - TDA

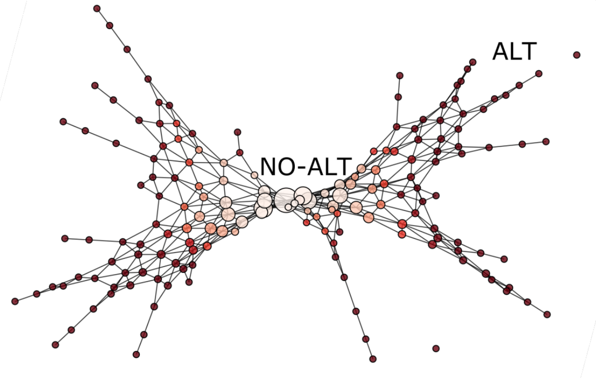


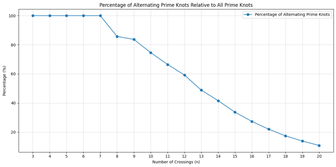
- Now live Ball mapper on knot data
- Play here https://dioscuri-tda.org/BallMapperKnots.html
  https://dustbringer.github.io/web–knot-invariant-comparison/

Data visualization

gives again many possible conjectures and comparisons

Alternating knots are actually easier than the general case (recalling the exponential decay theorem):

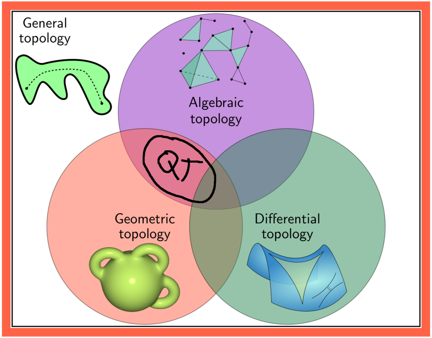




Most patterns that exists are probably to difficult to prove

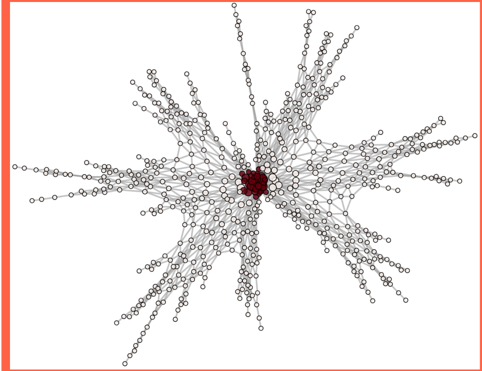
https://dustbringer.github.io/web–knot-invariant-comparison/

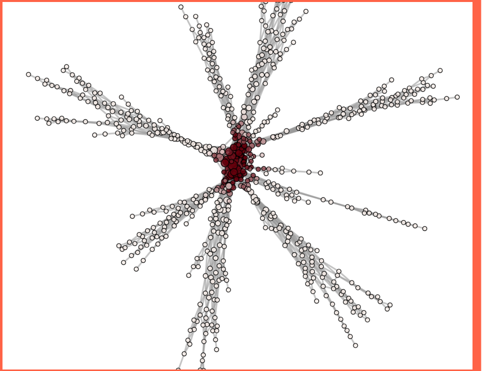
Big data and knots - compare



- Summary There is an infinite family of quantum invariants, “all” fail to detect knots fast and have superpolynomial runtime
- Essentially Before Jones we were missing knot invariants, after Jones we have too many and they are somewhat all them same
- Maybe what one should do instead is to compare them

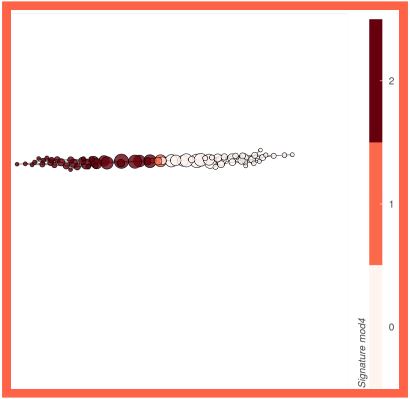
Big data and knots - compare





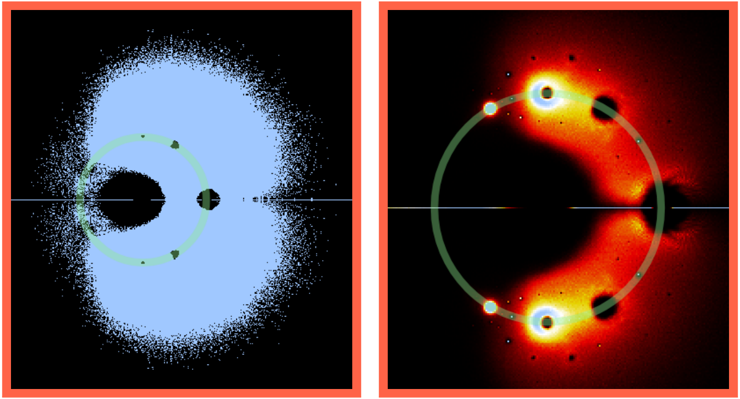
- Above Jones and its categorification (homology version)
- Categorification "=" pushing things further apart
- Comparing the invariants shows that they are related

Big data and knots - compare

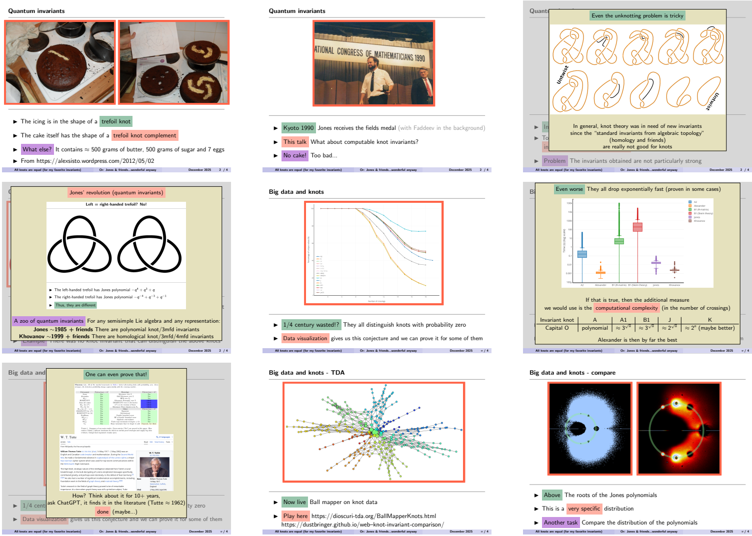


- Above Coloring of the Alexander invariant with the signature mode 4
- Signature = a traditional knot invariant (from homology)
- The eye catching conjecture is then easy to prove

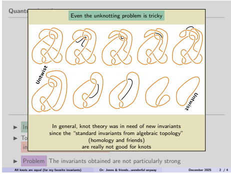
Big data and knots - compare

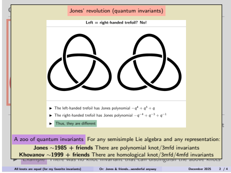


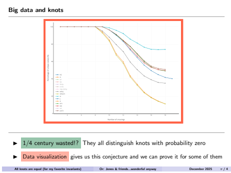
- Above The roots of the Jones polynomials
- This is a very specific distribution
- Another task Compare the distribution of the polynomials

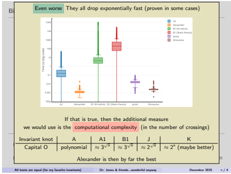


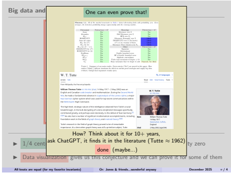
There is still much to do...

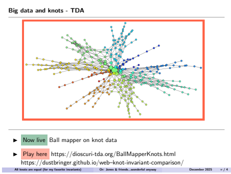


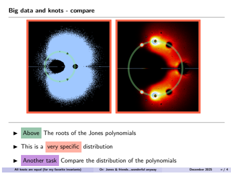












Thanks for your attention!