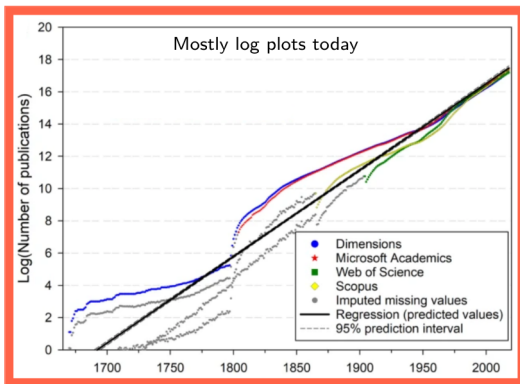


Counting in tensor products

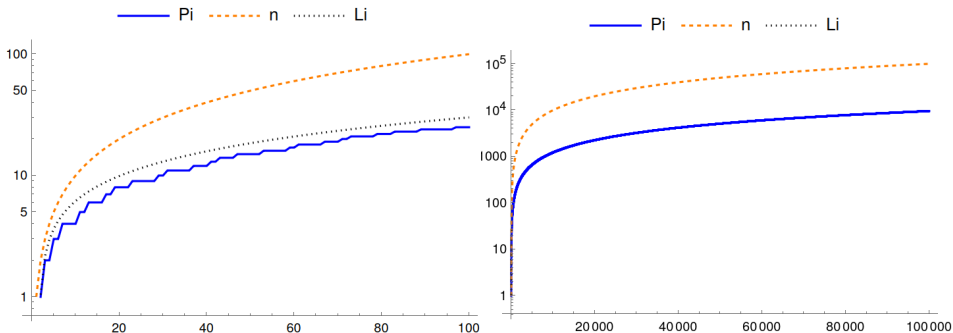
Or: Exponential growth everywhere

Accept ~~Change~~ what you cannot ~~change~~ accept



I report on work of Coulembier, Etingof, Ostrik, and many more

Let us not count!



- ▶ Prime number function $\pi(n) = \# \text{ primes } \leq n$
- ▶ Counting primes is very tricky as primes “pop up randomly”
- ▶ Question 1 What is the leading growth (of the number of primes)?
- ▶ Answer 1 There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

Let us not count!

Seriously, counting is difficult!

Limite x	Nombre γ		Limite x	Nombre γ	
	par la formule.	par les Tables.		par la formule.	par les Tables.
10000	1230	1230	100000	9588	9592
20000	2268	2263	150000	13844	13849
30000	3252	3246	200000	17982	17984
40000	4205	4204	250000	22035	22045
50000	5136	5134	300000	26023	25998
60000	6049	6058	350000	29961	29977
70000	6949	6936	400000	33854	33861
80000	7838	7837	Actually, #primes < 1000 = 1229...		
90000	8717	8713			

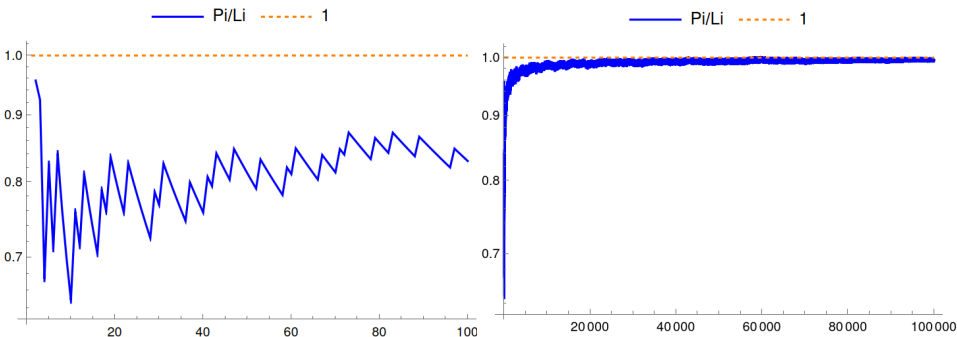
Legendre ~1808:
(for $n/(\ln n - 1.08366)$)

Gauss, Legendre and company counted primes up to $n = 400000$ and more

That took years (your iPhone can do that in seconds...humans have advanced!)

- ▶ **Question 1** What is the leading growth (of the number of primes)?
- ▶ **Answer 1** There are roughly $c(n) \cdot n$ for sublinear correction term $c(n)$

Let us not count!



- ▶ Asymptotically equal $f \sim g$ if $\lim_{n \rightarrow \infty} f(n)/g(n) \rightarrow 1$
- ▶ Logarithmic integral $\text{Li}(x) = \int_2^x 1/\ln(t) dt$
- ▶ Question 2 What is the growth (of the number of primes) asymptotically?
- ▶ Answer 2 We have $\pi(n) \sim n/\log(n) \sim \text{Li}(n)$

Riemann ~1859 calculates "the variance":

VII.

Ueber die Anzahl der Primzahlen unter einer
gegebenen Grösse.

(Monatsberichte der Berliner Akademie, November 1859.)

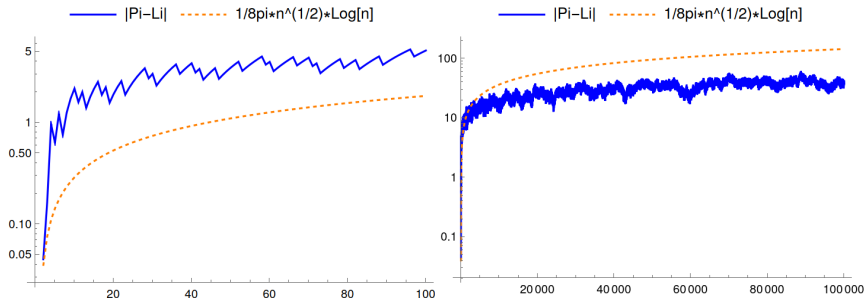
Durch Einsetzung dieser Werthe in den Ausdruck von $f(x)$ erhält man

$$f(x) = Li(x) - \sum^{\alpha} (Li(x^{\frac{1}{2} + \alpha i}) + Li(x^{\frac{1}{2} - \alpha i})) \\ + \int_x^{\infty} \frac{1}{x^2 - 1} \frac{dx}{x \log x} + \log \xi(0),$$

wenn in \sum^{α} für α sämtliche positiven (oder einen positiven reellen Theil enthaltenden) Wurzeln der Gleichung $\xi(\alpha) = 0$, ihrer Grösse nach geordnet, gesetzt werden. Es lässt sich, mit Hülfe einer genaueren Discussion der Function ξ , leicht zeigen, dass bei dieser Anordnung der Werth der Reihe

f is essentially the prime counting function π

Let us not count!



- ▶ Asymptotically equal does not imply that the difference is good
- ▶ $|f(n) - g(n)|$ is a measurement of how good the approximation is
- ▶ Question 3 What is variance from the expected value ($Li(n)$)?
- ▶ Conjectural answer 3 We have $|\pi(n) - Li(n)| \in O(n^{1/2} \log n)$ or $|\pi(n) - Li(n)| \leq \frac{1}{8\pi} n^{1/2} \log n$ (for $n \geq 2657$)

Let us not count!

— $|P_i - Li|$ - - - $1/8\pi \cdot n^{1/2} \cdot \text{Log}[n]$

— $|P_i - Li|$ - - - $1/8\pi \cdot n^{1/2} \cdot \text{Log}[n]$

What to expect from not counting



Leading growth



Asymptotic



"Variance"

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Let us not count!



- Γ = something that has a tensor product (more details later)
- \mathbb{K} = any ground field, V = any fin dim Γ -rep
- **Problem** Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Examples of what Γ could be

Any finite group, monoid, semigroup

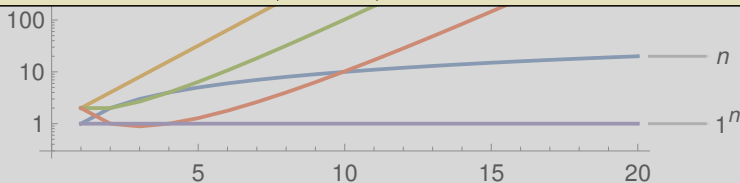
Symmetric groups, alternating groups, cyclic groups, the monster, $GL_N(\mathbb{F}_{p^k})$, ...

Actually **any** group, monoid, semigroup

$GL_N(\mathbb{C})$, $GL_N(\mathbb{R})$, $GL_N(\overline{\mathbb{F}_{p^k}})$, symplectic, orthogonal, braid groups, Thompson groups, ...

Super versions

$GL_{M|N}$, $OSP_{M|2N}$, periplectic, queer, ...



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100

Examples (that we will touch later)

Up to some slight change of setting we could also include:

Fusion categories or even finite additive Krull–Schmidt monoidal categories

$\mathbf{Proj}(G, \mathbb{K})$, $\mathbf{Inj}(G, \mathbb{K})$, semisimpl. of quantum group reps, Soergel bimodules of finite type, ...

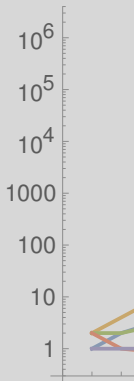
General additive Krull–Schmidt monoidal categories up to one condition (given later)

$\mathbf{Rep}(GL_n)$ and friends, quantum group reps, Soergel bimodules of affine type, ...

Most importantly, **your** favorite example might be included on this list

► Problem Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Let us not count!



$$\dim V)^n$$

$$\dim V)^n$$

$$n$$

$$\dim V)^n$$

$$n^2$$

Let us pause for a second...the setting is way to general!

Decomposing $V^{\otimes n}$ for an arbitrary group is not happening

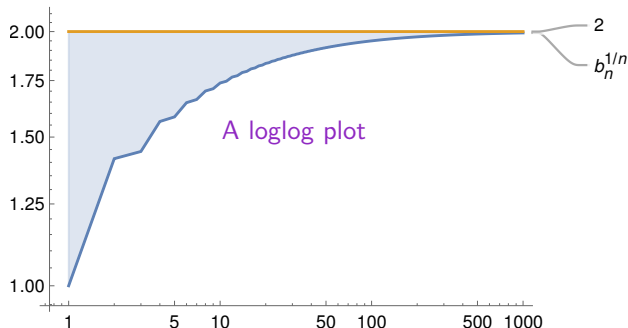
► Γ = som

Better: Let us answer a not counting question!

► \mathbb{K} = any ground field, V = any fin dim Γ -rep

► **Problem** Decompose $V^{\otimes n}$; note that $\dim_{\mathbb{K}} V^{\otimes n} = (\dim_{\mathbb{K}} V)^n$

Leading growth for “groups”

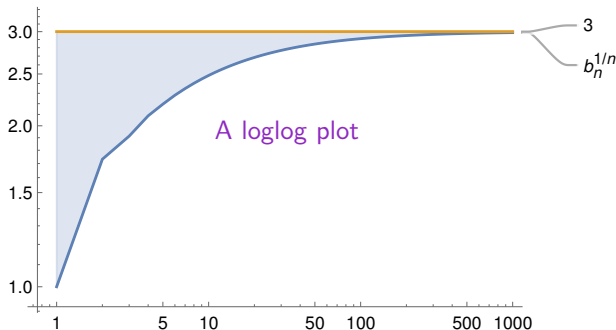


- $b_n = b_n^{\Gamma, V}$ = number of indecomposable summands of $V^{\otimes n}$ (with multiplicities)
- **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \mathbb{C}^2$, then

$$\{1, 1, 2, 3, 6, 10, 20, 35, 70, 126, 252\}, \quad b_n \text{ for } n = 0, \dots, 10.$$

$\lim_{n \rightarrow \infty} \sqrt[n]{b_n}$ seems to converge to $2 = \dim_{\mathbb{C}} V$: $\sqrt[1000]{b_{1000}} \approx 1.99265$

Leading growth for “groups”



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- ▶ **Example** $\Gamma = SL_2$, $\mathbb{K} = \mathbb{C}$, $V = \text{Sym } \mathbb{C}^2$, then

$$\{1, 1, 3, 7, 19, 51, 141, 393, 1107, 3139, 8953\}, \quad b_n \text{ for } n = 0, \dots, 10.$$

$$\lim_{n \rightarrow \infty} \sqrt[n]{b_n} \text{ seems to converge to } 3 = \dim_{\mathbb{C}} V: \quad \sqrt[1000]{b_{1000}} \approx 2.9875$$

Observation 1

Whatever is true for SL_2 over \mathbb{C} is true in general, right?

So let us come back to the general setting:

Γ = affine semigroup superscheme

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Observation 2

$$b_n b_m \leq b_{n+m} \Rightarrow$$

$$\beta = \lim_{n \rightarrow \infty} \sqrt[n]{b_n}$$

is well-defined by a version of Fekete's Subadditive Lemma

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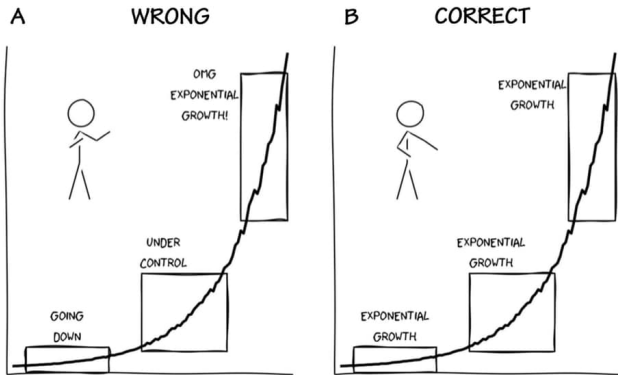
Observation 3

$$1 \leq \beta \leq \dim_{\mathbb{K}} V$$

$$\beta = 1 \Leftrightarrow V^{\otimes n} \text{ for } n \gg 0 \text{ is 'one block'}$$

$$\beta = \dim_{\mathbb{K}} V \Leftrightarrow \text{summands of } V^{\otimes n} \text{ for } n \gg 0 \text{ are 'essentially one-dimensional'}$$

Leading growth for “groups”



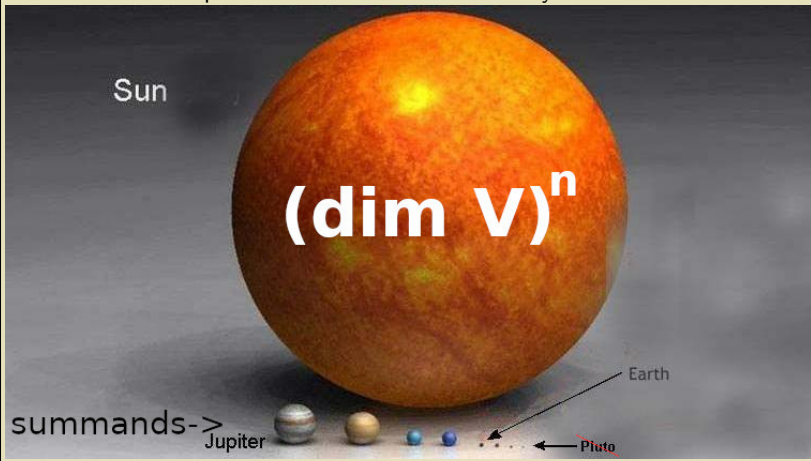
Coulembier–Ostrik ~2023 We have

$$\beta = \lim_{n \rightarrow \infty} \sqrt[n]{b_n} = \dim_{\mathbb{K}} V$$

Exponential growth is scary

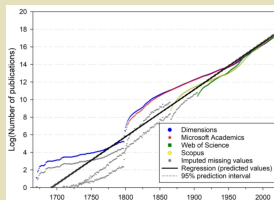
Roughly what this shows is “ $b_n \sim c(n) \cdot (\dim_{\mathbb{K}} V)^n$ ” for subexponential $c(n)$

In other words, compared to the size of the exponential growth of $(\dim_{\mathbb{K}} V)^n$
all indecomposable summands are ‘essentially one-dimensional’



On the next slide there is a formula of the form

$$\underbrace{b_n}_{b(n)} \sim \underbrace{c(n) \cdot (\dim_{\mathbb{K}} V)^n}_{a(n)}$$



We will explore the formula by examples
so no need to memorize it

The take away messages are:

The formula is completely explicit and works in quite some generality specified later

It only depends on eigenvalues and eigenvectors associated to a matrix

The assumptions on the next slide are not necessary
but make the formula look nicer

The recurrent case – everything goes

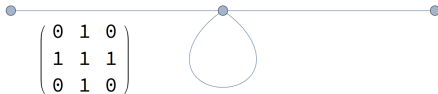
- ▶ Take a finite based $\mathbb{R}_{\geq 0}$ -algebra R with basis $C = \{c_0, \dots, c_{r-1}, \dots\}$
- ▶ Assume that R is the Grothendieck ring of our starting category
- ▶ For $a_i \in \mathbb{R}_{\geq 0}$, the action matrix M of $c = a_0 \cdot c_0 + \dots + a_{r-1} \cdot c_{r-1} \in R$ is the matrix of left multiplication of c on C
- ▶ Assume that M has a leading eigenvalue λ of multiplicity one; all other eigenvalues of the same absolute value are $\exp(k2\pi i/h)\lambda$ for some h
- ▶ Denote the right and left eigenvectors of M for λ and $\exp(k2\pi i/h)\lambda$ by v_i and w_i , normalized such that $w_i^T v_i = 1$
- ▶ Let $v_i w_i^T[1]$ denote taking the sum of the first column of the matrix $v_i w_i^T$
- ▶ The formula $b(n) \sim a(n)$ we are looking for is ($\zeta = \exp(2\pi i/h)$)

$$b(n) \sim (v_0 w_0^T[1] \cdot 1 + v_1 w_1^T[1] \cdot \zeta^n + v_2 w_2^T[1] \cdot (\zeta^2)^n + \dots + v_{h-1} w_{h-1}^T[1] \cdot (\zeta^{h-1})^n) \cdot \lambda^n$$

- ▶ The convergence is geometric with ratio $|\lambda^{\text{sec}}/\lambda|$

The recurrent case – everything goes

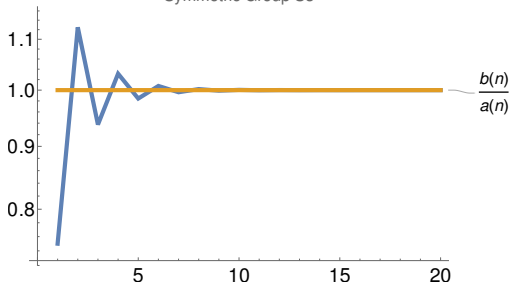
Symmetric group S_3 , $\mathbb{K} = \mathbb{C}$, V =standard rep



Example $\lambda = 2$, others=0, -1 , $v = w = 1/\sqrt{6}(1, 2, 1)$, $vw^T = \begin{pmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{pmatrix}$ and

$$a(n) = \frac{2}{3} \cdot 2^n$$

Symmetric Group S_3



The recurrent case – everything goes

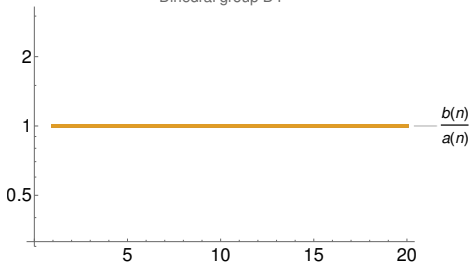
Dihedral group D_4 of order 8, $\mathbb{K} = \mathbb{C}$, V =defining rotation rep



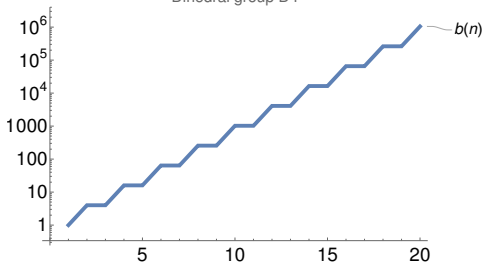
Example $\lambda = 2$, others $= -2, 0, 0, 0$, $v_\lambda = w_\lambda = 1/\sqrt{8}(1, 1, 1, 1, 2)$
 $v_{-2} = w_{-2} = 1/\sqrt{8}(-1, -1, -1, -1, 2)$ and

$$a(n) = \left(\frac{3}{4} + \frac{1}{4}(-1)^n\right) \cdot 2^n$$

Dihedral group D4



Dihedral group D4



The recurrent case – everything goes

Dihedral group D_4 of order 8, $\mathbb{K} = \mathbb{C}$, V =defining rotation rep

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

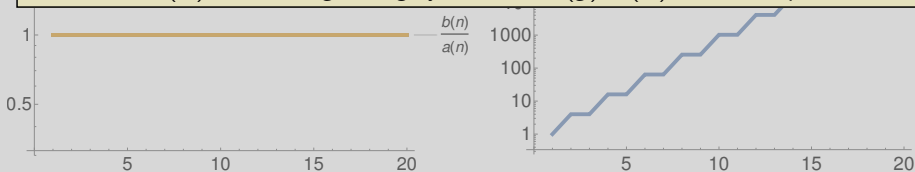


Example (general finite group, $\mathbb{K} = \mathbb{C}$, V =any faithful G -rep)

In this case we have a general formula:

$$a(n) = \left(\frac{1}{\#G} \sum_{g \in Z_V(G)} \left(\sum_{L \in S(G)} \omega_L(g) \dim_{\mathbb{C}} L \right) \cdot \omega_V(g)^n \right) \cdot (\dim_{\mathbb{C}} V)^n$$

$Z_V(G)$ =elements g acting by a scalar $\omega_V(g)$; $S(G)$ =set of simples

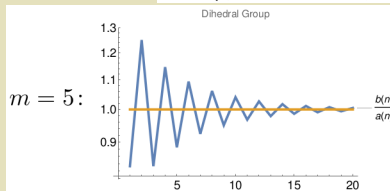


Example (continued)

Symmetric group S_m $a(n) = \left(\sum_{k=0}^{m/2} 1/((m-2k)!k!2^k) \right) \cdot (\dim_{\mathbb{C}} V)^n$

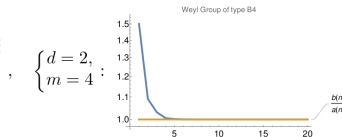
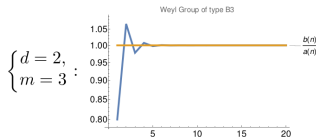
Dihedral group D_m of order $2m$

$$a(n) = \begin{cases} \frac{m+1}{2m} \cdot 2^n & \text{if } m \text{ is odd,} \\ \frac{m+2}{2m} \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is odd,} \\ \left(\frac{(m+2)}{2m} \cdot 1 + \frac{1}{m} \cdot (-1)^n \right) \cdot 2^n & \text{if } m \text{ is even and } m' \text{ is even.} \end{cases}$$

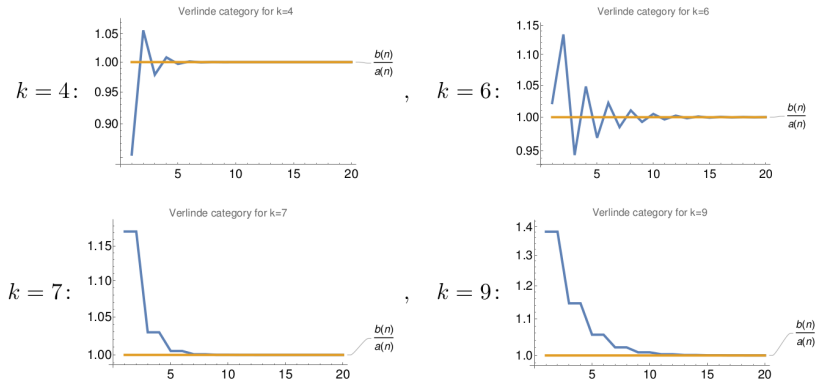


Complex reflection group $G(d, 1, m)$

$$\left\{ \begin{matrix} d=1, \\ m=3 \end{matrix} : a(n) = \frac{2}{3} \cdot 3^n, \right. \quad \left\{ \begin{matrix} d=2, \\ m=3 \end{matrix} : a(n) = \frac{5}{12} \cdot 3^n, \right. \quad \left\{ \begin{matrix} d=2, \\ m=4 \end{matrix} : a(n) = \left(\frac{19}{96} \cdot 1 + \frac{1}{32} \cdot (-1)^n \right) \cdot 4^n$$



The recurrent case – everything goes



Example For the SL_2 Verlinde category over \mathbb{C} at level k and $V = \text{gen. object}$:

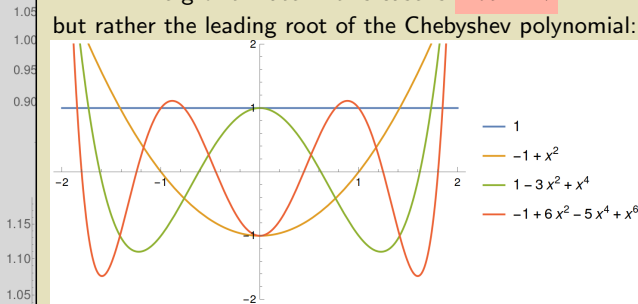
$$a(n) = \begin{cases} \frac{[1]_q + \dots + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot (2 \cos(\pi/(k+1)))^n & \text{if } k \text{ is even,} \\ \left(\frac{[1]_q + \dots + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot 1 + \frac{[1]_q - [2]_q + \dots - [k-1]_q + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot (-1)^n \right) \cdot (2 \cos(\pi/(k+1)))^n & \text{if } k \text{ is odd.} \end{cases}$$

The recurrent

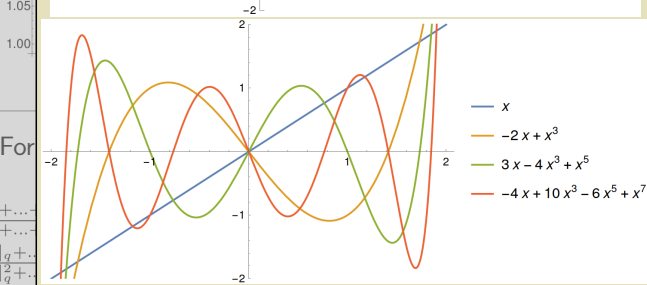
Example (continued)

The growth rate in this case is not in \mathbb{N}
but rather the leading root of the Chebyshev polynomial:

$k = 4$:



$k = 7$:



Example For

$$a(n) = \begin{cases} \frac{[1]_q + \dots + [1]_q}{[1]_q^2 + \dots + [1]_q^2} \\ \left(\frac{[1]_q + \dots + [1]_q}{[1]_q^2 + \dots + [1]_q^2} \right)^n \end{cases}$$

$$\frac{b(n)}{a(n)}$$

20

$$\frac{b(n)}{a(n)}$$

20

n. object:

if k is even,

if k is odd.

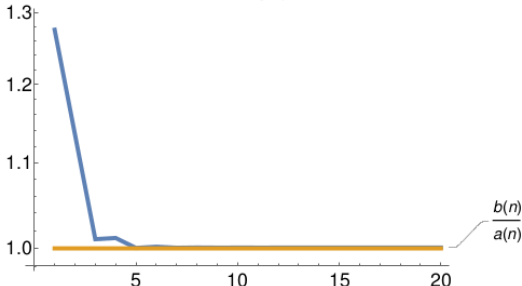
The recurrent case – everything goes

Example (continued)

Here is the SL_3 Verlinde category over \mathbb{C} at level $k = 4$ and $V = \text{gen. object}$:

$$k = 4: a(n) = \frac{1}{7} \left(2 + 2 \cos \left(\frac{3\pi}{7} \right) \right) \cdot \left(1 + 2 \cos \left(\frac{2\pi}{7} \right) \right)^n,$$

SL3 Verlinde category for k=4

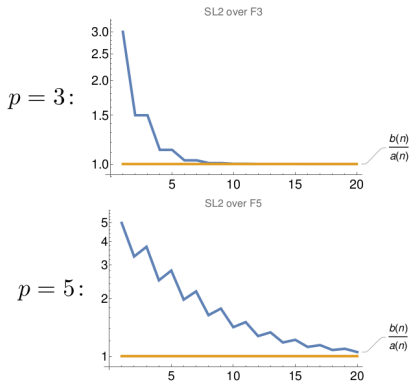


$k = 4:$

Koornwinder polynomials make their appearance

$$\left(\left(\frac{[1]_q + \dots + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot 1 + \frac{[1]_q + \dots + [k]_q}{[1]_q^2 + \dots + [k]_q^2} \cdot (-1)^n \right) \cdot (2 \cos(\pi/(k+1))) \right)^n \text{ if } k \text{ is odd.}$$

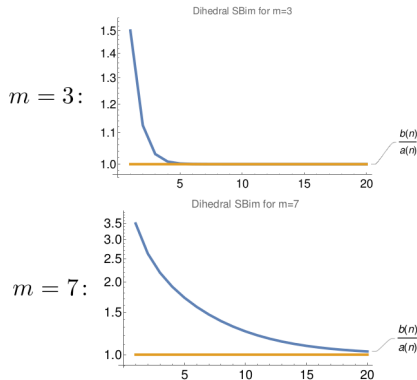
The recurrent case – everything goes



Example For $\text{SL}_2(\mathbb{F}_p)$, $\mathbb{K} = \mathbb{F}_p$ and $V = \mathbb{F}_p^2$ we get:

$$a(n) = \left(\frac{1}{2p-2} \cdot 1 + \frac{1}{2p^2-2p} \cdot (-1)^n \right) \cdot 2^n$$

The recurrent case – everything goes

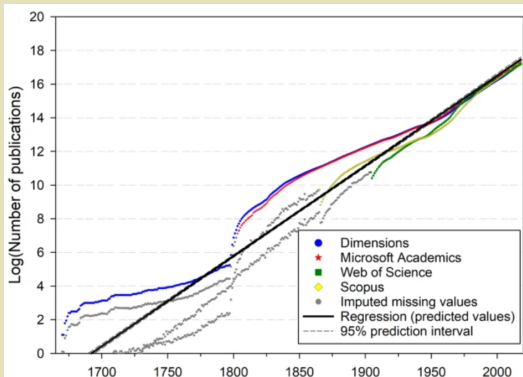


Example For dihedral Soergel bimodules of D_m , $\mathbb{K} = \mathbb{C}$ and $V = B_{st}$ we get:

$$a(n) = \frac{1}{2m} \cdot 4^n$$

The recurrent case – everything goes

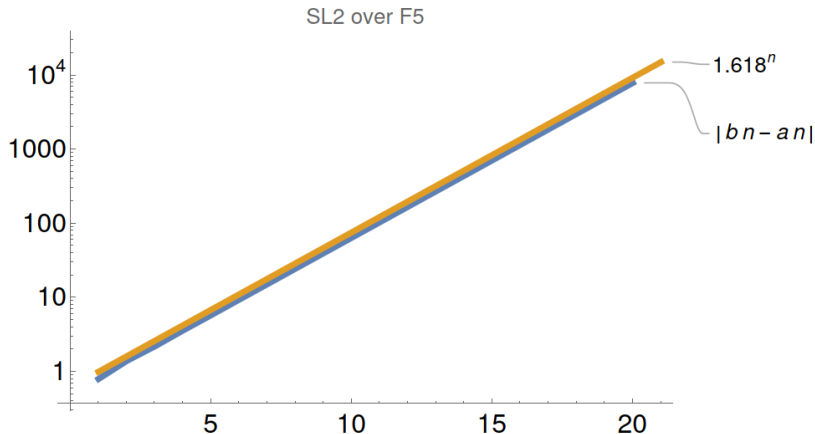
Observe that the growth of $b(n)$ is always exponential



Example For linear Sierpinski triangles of D_m , \mathbb{R}^m and $v \in D_{st}$ we get:

$$a(n) = \frac{1}{2m} \cdot 4^n$$

The recurrent case – everything goes



- ▶ The **variance** is given by $(\lambda_{\text{sec}})^n$ (second largest EV)
- ▶ **Example** Above for $\text{SL}_2(\mathbb{F}_5)$, $\mathbb{K} = \mathbb{F}_5$ and $V = \mathbb{F}_5^2$, λ_{sec} =golden ratio

The recurrent case – everything goes

VORLESUNGEN ÜBER DAS IKOSAEDER

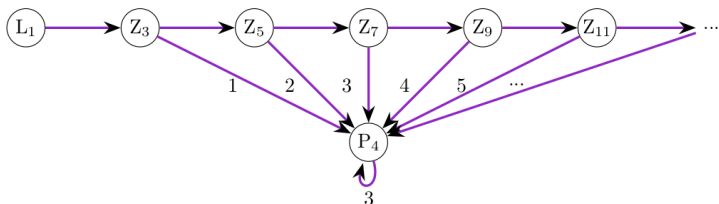
UND DIE
AUFLÖSUNG
DER
GLEICHUNGEN VOM FÜNFTEN GRADE
VON

FELIX KLEIN, 1884
U. S. PROFESSOR DER MATHEMATIK AN DER UNIVERSITÄT GIESSEN.

VON KLEIN

Offenbar umfasst unsere neue Gruppe von der Identität abgesehen nur Operationen von der Periode 2, und es ist zufällig, dass wir eine dieser Operationen an die Hauptaxe der Figur, die beiden anderen an die Nebenaxe geknüpft haben. Dementsprechend will ich die Gruppe mit einem besonderen Namen belegen, der nicht mehr an die Dieder-configuration erinnert, und sie als Vierergruppe benennen.

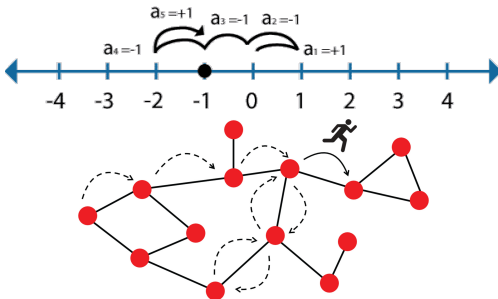
... die man



Example For the Klein four group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, $\mathbb{K} = \overline{\mathbb{F}_2}$ and $V = Z_3 = 3d$ inde. we get:

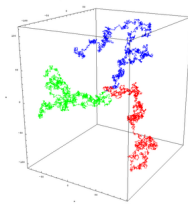
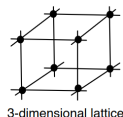
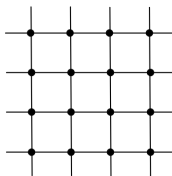
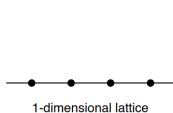
$$b_n \sim 3^n$$

The recurrent case – everything goes



- ▶ We randomly walk on some (connected) graph = at each step choose the next step/edge randomly but equally likely “coin flip walk”
- ▶ Question How often do we visit a vertex?
- ▶ Recurrent := We will hit every point infinitely often with $P(\text{robability})=1$
- ▶ Example Every (random walk on a) finite graph is recurrent

The recurrent case – everything goes



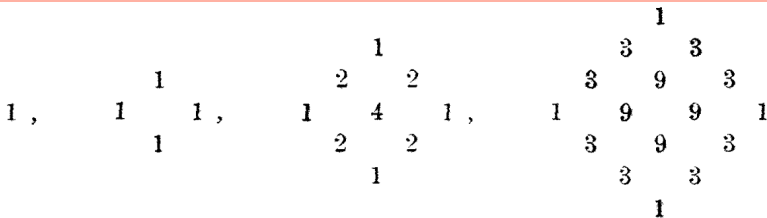
- ▶ **Pólya** ~ 1921 \mathbb{Z}^d is recurrent/transient $\Leftrightarrow d \leq 2/d > 2$
- ▶ A drunkard will find their way home, but a drunken bird may get lost forever
- ▶ **Transient** := We will hit every point finitely often with $P(\text{robability})=1$

Pólya ~1921

Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Straßennetz.

Von

Georg Pólya in Zürich.



- ▶ A drunkard will find their way home, but a drunken bird may get lost forever
- ▶ **Transient** := We will hit every point finitely often with $P(\text{robability})=1$

Every graph is either recurrent or transient



This is an instance of a 0-1-theorem :
a lot of properties hold with $P=0$ or $P=1$ but $0 < P < 1$ rarely appears

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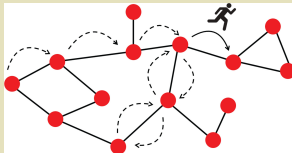


This is an instance of a 0-1-theorem :
a lot of properties hold with $P=0$ or $P=1$ but $0 < P < 1$ rarely appears

Perron ~1907, Frobenius ~1912, Vere-Jones ~1967, etc.

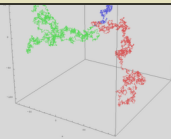
- ▶ F
- ▶ A The previous eigenvalue strategy applies to (positively) recurrent settings : rever
- ▶ For $b_n(V)$ take the fusion graph for V and check whether it is recurrent 1

Examples of recurrent growth problems



Easy If one has finitely many indecomposables

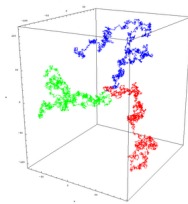
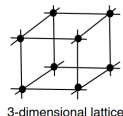
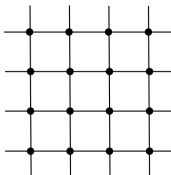
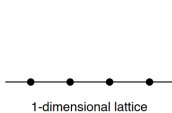
Coulembier–Etingof–Ostrik ~2023 V is an object of a finite tensor categories



Perron ~1907, Frobenius ~1912, Vere-Jones ~1967, etc.

The previous eigenvalue strategy applies to (positively) recurrent settings : rever
For $b_n(V)$ take the fusion graph for V and check whether it is recurrent 1

The recurrent case – everything goes



- **Pólya** ~ 1921 $b_n(V)$ for V a faithful compl. decomposable Γ -rep in char zero is recurrent $\Leftrightarrow \Gamma$ is virtually \mathbb{Z}^d for $d \in \{0, 1, 2\}$
- **Virtually** means we allow extensions by finite groups

Biané ~1993, Coulembier–Etingof–Ostrik ~2023

showed that surprisingly (not recurrent!)

for complex fin dim simple Lie algebras (\mathfrak{sl}_n +friends) in char zero

one can still answer the three growth questions

2.2. THÉORÈME :

$$m(\lambda, E^{\otimes n}) = 0 \quad \text{si } \lambda \notin nP(E) + Q(E)$$

$$= \frac{\prod_{\alpha \in R_+} q^*(\alpha, \rho)}{\text{vol}_q(\mathfrak{h}_{\mathbb{R}}/Q^\vee)} \frac{k(E) d(E)^n}{(2\pi)^{l/2} n^{m/2}} d(\lambda) \left(e^{-(q^*(\lambda+\rho)/2)n} + O\left(\frac{1}{n}\right) \right) \text{ sinon}$$

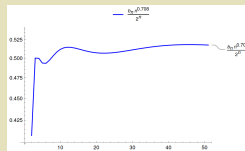
Le terme $O(1/n)$ est uniforme en $\lambda \in P_{++}$, et $\text{vol}_q(\mathfrak{h}_{\mathbb{R}}/Q^\vee)$ désigne la mesure pour dx d'un domaine fondamental du réseau Q^\vee .

Exp. fac. $d(E)^n = (\dim_{\mathbb{C}} E)^n$, subexp. fac. $n^{\#\text{pos. roots}/2}$, some scalar, variance

Char p is difficult, even for SL_2

The subexp. factor has transcendental power (fractals!)

the “scalar function” is highly oscillating, etc.



► Pólya ~1

char zero

► Virtually

Let us not count!

(Surprisingly, counting is difficult!)

Line x	Number y to be formed by the Yabos	Line x	Number y to be formed by the Yabos
10000	10000	10000	10000
9999	9999	9999	9999
9998	9998	9998	9998
9997	9997	9997	9997
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9995	9995	9995	9995
9994	9994	9994	9994
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9921	9921	9921	9921
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9919	9919	9919	9919
9918	9918	9918	9918
9917	9917	9917	9917
9916	9916	991	

The recurrent case – everything goes

UNDETERMINED

FREE AND ISOLATED

UNDETERMINED

CLASSIFICATION FOR PERIOD FOUR

1884

Obwohl selbst unsere von Frege zur Disposition algorithmische Sprache nicht in der Lage ist, in völliger, zum vor-mechanischen Zeitalter gehörender Weise, die beiden ersten der oben besprochenen Aufgaben zu lösen, so ist die Frage nach der Möglichkeit der Lösung der zweiten Aufgabe durch die Frege'sche Sprache nicht weniger interessant, als die Frage nach der Lösung der ersten Aufgabe, und die sich aus der Dispositionsalgorithmik ergibt, und nach dem **Interpretationssatz**.

Example for the Klein four group $Z_2 \times Z_2 \times Z_2$, $K = \overline{K}_2$ and $V = V_1$ (3d id): we get:

How good is this not counting?

Riemann – 1859 calculates “the variance”.

VII.

Vorleser der Anzahl der Primzahlen unter einer gegebenen Grösse.

(Humboldtstr. der Berliner Akademie, Sommer 1859)

Deutsch: Erwartung: Anzahl Warte in der Anzahl von N ist $\pi(N)$ ist

$$\rho(x) = Li(x) - \frac{1}{2} Li(x^{1/2}) + Li(x^{1/3}) - \dots$$

$$+ \frac{1}{2} \left(\frac{1}{x^{1/2}} - \frac{1}{x^{1/4}} + \frac{1}{x^{1/3}} + \log 100 \right),$$

where is 2×10^6 a standard modulus (also some positive terms). That calculation is the standard $Li(x)$. Are you well-grounded, good? Yes. So last slide, not HRR clear your chosen divergence the Function $\rho(x)$ does. And last slide: the change the $\rho(x)$ to $\rho(x) + \frac{1}{2} \log 100$.

ρ is essentially the prime counting function π .

Exponential growth is scary

Roughly what this shows is $\|u_n - c(x)\| \sim (\dim V)^n$ for subexponential $c(x)$

In other words, compared to the size of the exponential growth of $(\dim V)^n$ all subexponential summands are "essentially one-dimensional"

Sun

$(\dim V)^n$

summands > 1/n

1 cm

Source: 2008

The recurrent case – everything goes

- ▶ We **randomly walk** on some **connected** graph G : at each step choose the next step *edge* randomly (uniformly) – “coin flip walk”
- ▶ **Question:** How often do we visit a vertex?
- ▶ **Recurrence** – We will hit every point **infinitely often** with $P(\text{revisiting}) = 1$
- ▶ **Example** Every (random walk on a) finite graph is recurrent

Let us not count!

naive: $\Theta(n^2)$ naive: $\Theta(n^2 \log n)$ naive: $\Theta(n^2 \log n)$

What to expect from not counting

EASY Leading growth

MEDIUM Asymptotic


HARD "Variance"

► **Naïveté** is a virtue in variance (not the expected value $E(\chi_i^2)$)


► **Conjectural answer** We have $|e(n) - L(n)| \in O(n^{1/2} \log n)$ or $|e(n) - L(n)| \leq n^{o(1)} \log n$ (see e.g. [2003])

The recurrent case – everything goes


Symmetric group S_3 , $\mathbb{R} \in \mathbb{C}$, V : standard rep


$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$



Example $\lambda = 2$, others: $0, -1$, $v = w = 1/\sqrt{6}(1, 2, 1)$, $vw^T = \begin{pmatrix} 1/6 & 1/3 & 1/6 \\ 1/3 & 2/3 & 1/3 \\ 1/6 & 1/3 & 1/6 \end{pmatrix}$ and


$$a(n) = \frac{1}{3} \cdot 2^n$$


The recurrent case – everything goes

 1-dimensional lattice

 2-dimensional lattice

 3-dimensional lattice

 4-dimensional lattice

- **Pólya** – 1921 $\ln(V)$ for a faithful F -rep in **char zero** is recurrent **↔** F is virtually 2^d for $d \in [0, 1, 2]$
- **Virtually** means we allow extensions by finite groups

There is still much to do...

[illegible]

The recurrent case – everything goes

INDISTINGUISHABILITY

FREE INDISTINGUISHABILITY

QUESTION: How many partitions of n are there with no two consecutive integers?

ANSWER: $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269, 2178309, 3524558, 5702867, 9227465, 14930354, 24146823, 39285713, 63473144, 102703155, 167761390, 271476005, 439218619, 711486734, 1151955643, 1871148687, 3023215381, 4894371024, 7915526405, 12809897429, 20725413834, 33545281263, 54275178692, 88144682121, 142369850813, 230556832934, 372926815055, 603486647989, 980191486402, 1583654755491, 2566851402480, 4150543157979, 6717404550469, 10878355608440, 17595759660909, 28474105215368, 46069864876277, 74565624482186, 120639680097553, 195209294573829, 315848919661382, 511058609758935, 826867914422814, 1342926524184199, 2169794439937034, 3512721364121233, 5682515804058267, 9195306174185301, 14878021978243568, 24073328122428869, 38951349090672437, 63024377212916006, 101975695335348445, 164999072548264451, 266974767883580457, 431969840118928903, 708969012302189354, 1140968852421110257, 1849937864539939160, 3010906876961049414, 4860844741500988573, 7871751618462037987, 12632596360003026560, 20504347978465064547, 33137144338468091107, 53641741700471117667, 86779089038939181714, 140416830739409299281, 227158571440080480985, 367574450478519662699, 594733281917930838684, 962301852397440520383, 1557035134315371359067, 2519336986713302197751, 4076372119010743556835, 6595708005724045915892, 10672039524734748072727, 17267747530458793988619, 28039786555192839901346, 45306824085927587980965, 73346571616386381979584, 118656358173628221180930, 192002933480015809160495, 310659304653644191140079, 502662238133662410321014, 813318542787296601461509, 1315980780920958811601523, 2128643119054625412962537, 3444623861741922014564060, 5573266980662580826465597, 9017890092394502839029657, 14591516973056424865495254, 23609407065450926681524911, 38199297138503429520554565, 61800814103954355386050776, 100000321142457783906575341, 161799135250412140287126057, 261799456392869924193676802, 423799777535327608099802859, 685599108677785392293428961, 1109398485810655016487045820, 1804997644388442618680674781, 2914396850204127634973693642, 4719394494592782653654368423, 7633791135796909268628062065, 12353185630389691882502430488, 20086976766186601141156792553, 32440162396576503030759223041, 52526139126966194812861715594, 85066301523542795953620938635, 137592430650509398766482664229, 222658532176476194679344389823, 360250962827005990633006053464, 582909494993482185611627443303, 943162457169488176344633507767, 1525821952162970361956260951070, 2468984409332458548299894458837, 4002806361495428734656155410607, 6471790763657887193856049869444, 10474607125153305928512205280051, 16946397888811193122368354139495, 27421005013964500320880559419546, 44367392802775693443248913559041, 71788397816739193764131072978536, 116155789630704787184380086537581, 187944187447443980948511159497117, 304099977078148774132891246035653, 492044164515592755127402332533234, 796143141593741739260293578568887, 1288187306109334514387705911102021, 2084330447625076243648199489640908, 3372517753734410758035895400742929, 5456848160359486971724094889383837, 8829365914093897719810090290024766, 14286214074453384691534185179408603, 23115582234547282401344275469433440, 37391846308991179113154460648842149, 60507428543538461514498736118250789, 97899314852530640627652996767092938, 158406743396069102141807232885343727, 255806058238599742759460229603436666, 414212791634668844891267462490430603, 669918849873268547050727692093867579, 1084125641507937391942088154584308182, 1753044491381206239092815846678175761, 2837170132889143630934803999762483943, 4590214624270349870027619846450660704, 7427384757151556001962423846213044647, 12017599381421905872090043692663705351, 19444984138573461873017467538877350000, 31462583519995367874907511231540555351, 50907567658569269747924978870417855351, 82370151178562731621832490102288408702, 133277718837132001369757468973706264053, 215647870015691731117682459076094122804, 348925588852823732$

How good is this not counting?

Riemann – 1859 calculates “the variance”.

VII.

Unter der Annahme der Primzahlen verthe eine gewisse Grösse.

(Humboldt der Berliner Akademie, Sommer 1858)

Durch Extension dieser Warte in den Ausdruck von $\rho(x)$ erhält man

$$\rho(x) = Li(x) - \frac{1}{2} Li(x^{1/2}) + Li(x^{1/3}) - \dots$$

$$+ \frac{1}{p-1} \cdot \log x + \log 100,$$

was in 2^{ter} Form eine charakteristische Gleichung (oder eine positive reelle Gleichung) der Primzahlen $\rho(x)$ enthält. Aber diese Gleichung, gleich einer, die fast sich, mit Hülfe einer gewissen Extension der Function ρ selbst, sagen wir, dass die Lösung der Warte der Primzahlen $\rho(x)$ ist.

It is essentially the prime counting function π

Exponential growth is scary

Roughly what this shows is $\dim_c V^n \sim c^n (\dim_c V)^n$ for subsequential $c[n]$

In other words, compared to the size of the exponential growth of $(\dim_c V)^n$ all indecomposable summands are "essentially one-dimensional"

The diagram shows a large orange sphere representing the Sun. Below it, a small planet is shown with a scale bar indicating 1 cm. The text "(dim V)ⁿ" is written in large white letters over the Sun. Below the planet, the text "summands >" is written. The diagram illustrates the exponential growth of the dimension of the space of subspaces of a vector space V, compared to the size of the space V itself.

The recurrent case – everything goes

- ▶ We randomly **edge** on some **connected** graph = at each step choose the next step, edge randomly equally likely "coin flip walk"
- ▶ **Question:** How often do we visit a vertex?
- ▶ **Recurrent:** We will hit every point infinitely often with $P(\text{recurrence}) = 1$
- ▶ **Example:** Every (random walk on a) finite graph is recurrent

Let us not count!

naive: $O(n^2)$ | naive: $O(n^2 \log n)$ | naive: $O(n^2)$ | naive: $O(n^2 \log n)$

What to expect from not counting

EASY Leading growth

MEDIUM Asymptotic


HARD "Variance"

► **Question 1.** What is variance from the perspective value $\chi(\mathcal{G})$?


► **Conjectural answer 1** We have $|\pi(n) - L(n)| \leq O(n^{1/2} \log n)$ or $|\pi(n) - L(n)| \leq \frac{1}{2} n^{1/2} \log n$ (see e.g. [2015])

The recurrent case – everything goes

Symmetric group S_3 , $K \in \mathbb{C}$, V -standard rep

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$


Example $\lambda = 2$, others: $0, -1$, $v = w = \sqrt{3}/2 (1, 2, 1)$, $vw^T = \begin{pmatrix} 1/3 & 2/3 & 1/3 \\ 2/3 & 1/3 & 1/3 \\ 1/3 & 2/3 & 1/3 \end{pmatrix}$ and

$$a(n) = \frac{7}{2} \cdot 2^n$$


The recurrent case – everything goes

► **Pólya** – 1921 $\ln(V)$ for a faithful f -rep in **other zero** is recurrent **ex** f is virtually $2d^*$ for $d \in \{0, 1, 2\}$

► **Virtually** means an affine extension by finite groups

Thanks for your attention!