What is...tropical geometry - part 21?

Or: Tropical linear algebra 5 - Tropical matrix ranks

Matrix ranks, part 1

```
In[1]:= a := {{1}, {2}, {3}};
        b := {{4, 5, 6}};

In[5]:= a // MatrixForm
        b // MatrixForm
        a.b // MatrixForm
```

rank-1 matrix = column x row vector:

Out[5]//MatrixForm=

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Out[6]//MatrixForm=

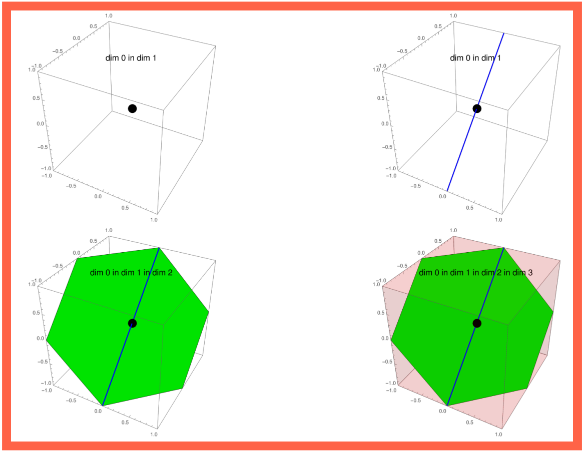
$$\begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$$

Out[7]//MatrixForm=

$$\begin{pmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{pmatrix}$$

- Matrix rank = one of the most important concepts of linear algebra
- Rank = smallest r such that $A = \sum_{i=1}^{r}$ rank-1 matrices
- The same definition applies for tropical matrices (called Barvinok rank)

Matrix ranks, part 2



- Matrix rank = one of the most important concepts of linear algebra
- Rank = smallest dimension of any linear space containing the column vectors
- The same definition applies for tropical matrices (called Kapranov rank)

Matrix ranks, part 3

$$\begin{bmatrix} 1 & 4 & \square \\ \square & \square & \square \\ -1 & 9 & \square \end{bmatrix}$$

- Matrix rank = one of the most important concepts of linear algebra
- Rank = largest r such that A has a nonsingular r-by-r minor
- The same definition applies for tropical matrices (called tropical rank)

For completeness: A formal statement

For every matrix A with entries in the tropical semiring:

Tropical rank of A $\leq$ Kapranov rank of A $\leq$ Barvinok rank of A

and these inequalities can be strict

- They are not the same ☹
- They are not the same ☺(see next slide)
- Example of Barvinok rank two:

$$M = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 1 & 0 \\ 2 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} \odot (0,4,2) \oplus \begin{pmatrix} 3 \\ 0 \\ 3 \end{pmatrix} \odot (2,1,0).$$

Picture from "Introduction to Tropical Geometry" by Diane Maclagan and Bernd Sturmfels

Traveling salesperson problem (TSP) and tropical matrix ranks

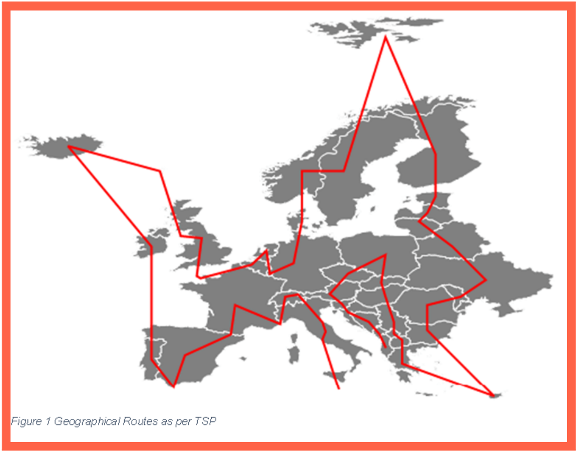


Figure 1 Geographical Routes as per TSP

- All of these have different applications ☺
- Example For fixed r, the TSP can be solved in polynomial time if the distance matrix is of Barvinok rank r; usually TSP is very difficult

Thank you for your attention!

I hope that was of some help.