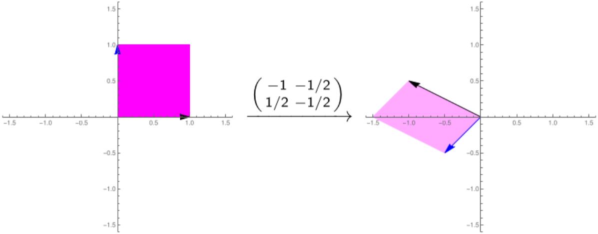
What is...tropical geometry - part 18?

Or: Tropical linear algebra 2 - determinants and optimization

Volumes of parallelograms

It is all about area

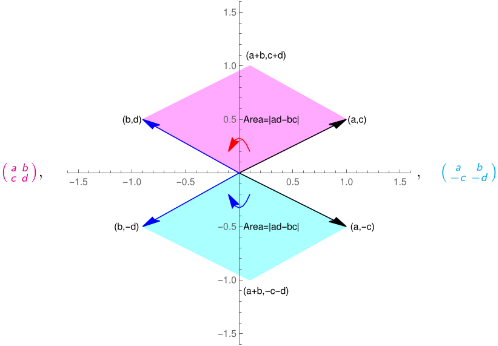


Wish. The determinant should be the area of matrix times unit square.

- Above The image of the unit square under the illustrated matrix
- Recall The area of the polytope on the right is measured by the determinant
- This is just the tip of the iceberg: determinants are everywhere

Well, there is a sign...

It is all about area – well, almost



So we let $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = -\det\begin{pmatrix} a & b \\ -c & -d \end{pmatrix} = ad - bc$

- Actually only the absolute value of the determinant measures the area
- More general, the determinant also takes orientation into account
- Under log we don't know signs

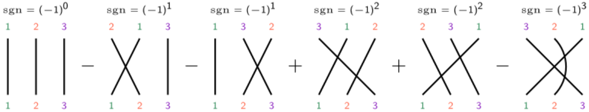
The determinant formula

Can you make this explicit?

$$\det(A) = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma_i}$$

Example. For $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$:

$\det(A) =$



$$= a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32} + a_{13}a_{21}a_{32} + a_{12}a_{23}a_{31} - a_{13}a_{22}a_{31}$$

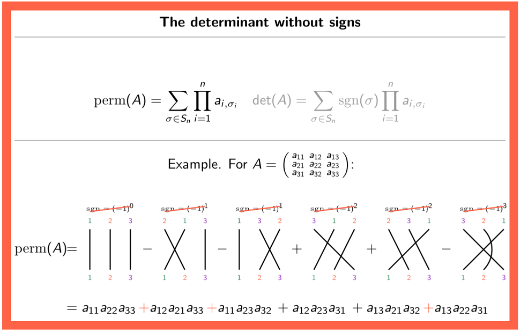
- Recall To compute the determinant take a signed sum
- The signed sum goes over all permutations
- Tropicalize Do the same tropically, but without sign

For completeness: A formal definition

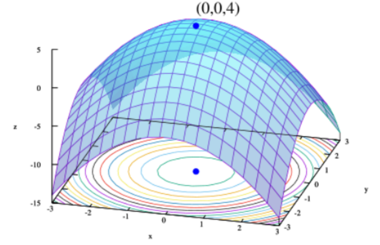
A tropical matrix $A = (a_{ij})_{i,j=1}^{n}$ has tropical determinant

$$\det(A) = \sum_{\text{permutations } \sigma} \prod_{i} a_{i\sigma(i)}$$

- This equals $\min_{\sigma}\{a_{1\sigma(1)} + \ldots + a_{n\sigma(n)}\}$
- This is more like the permanent



Optimization again



Graph of a surface given by $z = f(x, y) = -(x^2 + y^2) + 4$. The global maximum at $(x, y, z) = (0, 0, 4)$ is indicated by a blue dot.

- **Task** Suppose n workers must be assigned to n jobs, and a matrix A records the cost of training each worker for each job. How should the workers be assigned 1:1 so that the total training cost is minimized?
- **Answer** Any permutation that minimizes the tropical determinant works

Thank you for your attention!

I hope that was of some help.