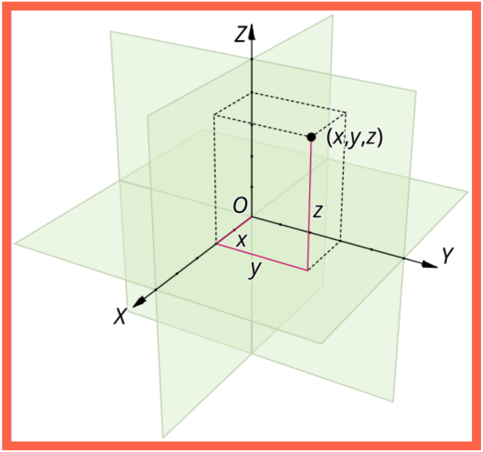
What is...tropical geometry - part 13?

Or: Non-Archimedean fields

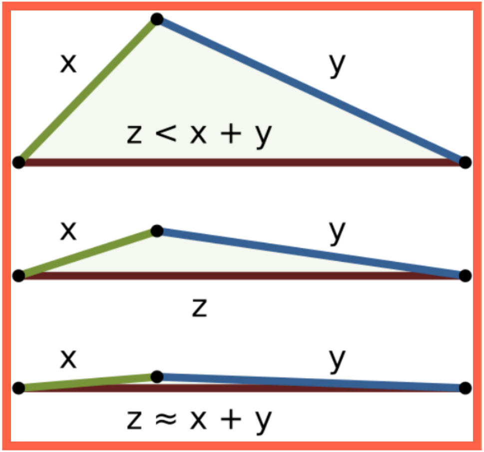
Normed spaces

$\mathbb{R}^3$:



- Norms provide a way to measure "length", essential for geometry
- Normed spaces allow us to define concepts like continuity etc.
- Norms mimic Euclidean space as above, but might be still different in flavor

Valuation to norm



- Anti-log A valuation $v\colon \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ induces a norm $|a| = \exp(-\mathrm{val}(a))$
- non-Archimedean We have $|a+b| \leq \max\{|a|, |b|\}$
- This is very different from the triangle inequality in $\mathbb{R}^3$

Example: p-adic numbers

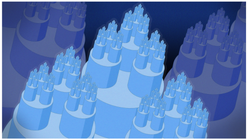
Enter, the theorem

p-adic integers $\mathbb{Z}_p$ and numbers $\mathbb{Q}_p$ exist via the equivalent definitions

(a) $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Q}_p$ is its field of fractions Algebra

(b) $\mathbb{Q}_p = \frac{(\text{Cauchy sequences in } \mathbb{Q} \text{ wrt } d)}{(\text{Nil sequences in } \mathbb{Q} \text{ wrt } d)}$ and $\mathbb{Z}_p$ is its ring of integers Analysis

$\mathbb{R} = \frac{(\text{Cauchy sequences in } \mathbb{Q} \text{ wrt } d)}{(\text{Nil sequences in } \mathbb{Q} \text{ wrt } d)}$ for the standard metric Analogy



Theorem (local-global). Let $f \in \mathbb{Q}[X_1, \dots, X_n]$ be nice

(a) If $f = 0$ holds in $\mathbb{Q}$, then it holds in $\mathbb{R}$ and $\mathbb{Q}_p$ for all p global$\to$local

(b) If $f = 0$ holds in $\mathbb{R}$ and $\mathbb{Q}_p$ for all p, then it holds in $\mathbb{Q}$ local$\to$global

- For $\mathbb{Q}$ the p-adic valuation is defined by $v_p(a) = \max\{n \in \mathbb{Z} : p^n \text{ divides } a\}$
- The induced p-adic norm is given by $|a|_p = p^{-\mathrm{val}_p(a)}$ (non-Archimedean)
- p-adic numbers $\mathbb{Q}_p$ form a complete field with respect to this norm

For completeness: A formal definition

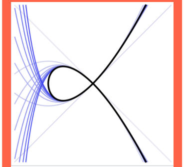
Consider a valuation $v : \mathbb{K} \to \mathbb{R} \cup \{\infty\}$ on a field $\mathbb{K}$; a norm on $\mathbb{K}$ is defined by:

$$|a| = \exp(-v(a)) \quad \text{for } a \neq 0, \quad |0| = 0$$

This norm satisfies the non-Archimedean property:

$$|a + b| \leq \max\{|a|, |b|\}$$

- Example (tropical) Rational functions (as a special case of the below)
- Example (Puiseux series) Puiseux series $\mathbb{C}\{\{x\}\}$ have a valuation given by the order of the lowest power of $x \Rightarrow$ non-Archimedean field



Truncated Puiseux expansions for the cubic curve $y^2 = x^3 + x^2$ at the double point $x = y = 0$. Darker colors indicate more terms.

Computable? Well...

- Puiseux series $=$ something of the form $\sum a_i x^{t_i}$ with $t_i \in \mathbb{Q}$
- Problem These often cannot be described by a finite amount of information
- For a computer fields like $\mathbb{Q}(x)$ are better

Thank you for your attention!

I hope that was of some help.