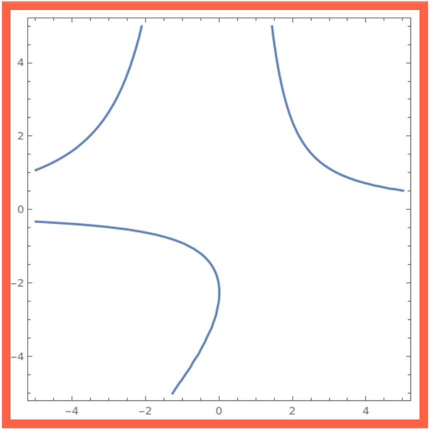
What is...tropical geometry - part 11?

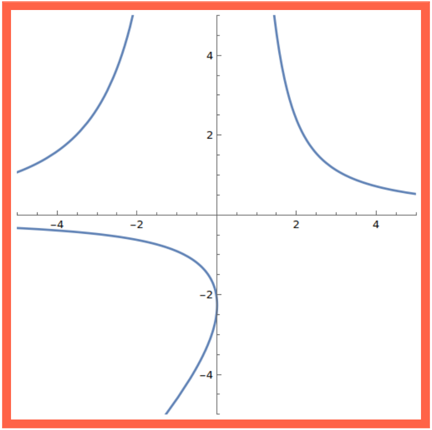
Or: Implicitization

Implicit plot



- Above A classical variety given by $x^3y^2 - x^2y^3 - 5x^2y^2 - 2x^2y - 4xy^2 - 33xy + 16y^2 + 72y + 81 = 0$
- Implicit plot = graph of an equation where the relationship between variables is given implicitly (not solved for one variable in terms of the others), such as $p(x, y) = 0$; this is the traditional way to describe varieties

Parametric plot



- **Above** The previous variety given by $t \mapsto ((t^3+4t^2+4t)/(t^2-1), (t^3-t^2-t+1)/t^2)$
- **Parametric plot** = graph where the coordinates are defined by one or more parameters, typically expressing x and y as functions of a third variable (often time), like $(x(t), y(t))$

Going from implicit to parametric and back

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GroebnerBasis[{X^2+Y^3-1, X-Y^2-X*Y}, {X, Y}]
{-1+2 Y-Y^2+Y^3-Y^4+Y^5, -1+X+Y-Y^2-Y^4}
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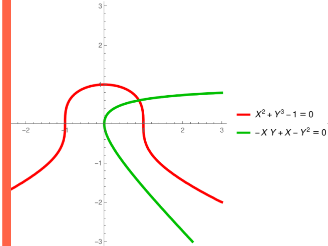
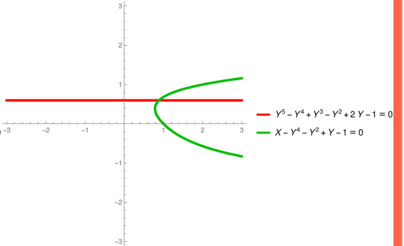




$$\begin{cases} X^2+Y^3-1=0 \\ -XY+X-Y^2=0 \end{cases} \rightsquigarrow \begin{cases} Y^5-Y^4+Y^3-Y^2+2Y-1=0 & \text{Only in } Y \\ X-Y^4-Y^2+Y-1=0 & \text{Trivial for fixed } Y \end{cases}$$

- Gröbner bases and resultants allow us to go between the two
- Problem These are not easy to compute
- Question Can we do this tropically?

For completeness: A formal statement (say for curves)

Given two rational functions ϕ_1, ϕ_2 as products of linear factors:

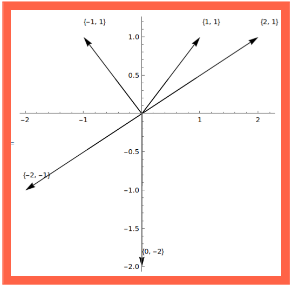
$$\phi_1(t) = (t-\alpha_1)^{u_1} \cdots (t-\alpha_m)^{u_m}, \phi_2(t) = (t-\alpha_1)^{v_1} \cdots (t-\alpha_m)^{v_m}$$

(i) Only the integer exponents u_i, v_i matter, not the α_i

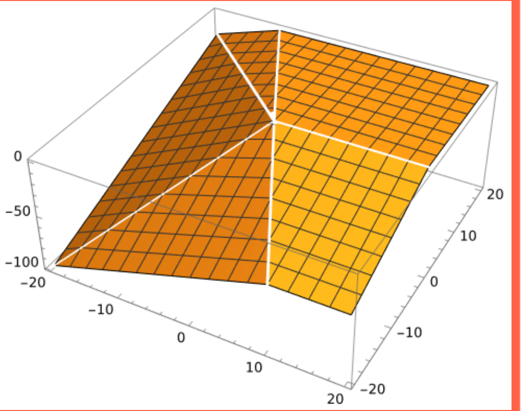
(ii) Define $u_0 = -u_1 - ... - u_m$, $v_0 = -v_1 - ... - v_m$

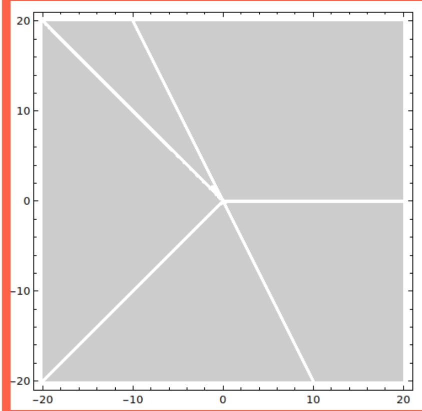
(iii) Form the vector collection: $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, ..., \begin{pmatrix} u_m \\ v_m \end{pmatrix}$

- Tropical curve Spanned by these $m+1$ rays in $\mathbb{R}^2$
- The tropical curve determined by the exponents of the previous example:



Wait, where is the tropical curve?





- Above = the tropicalization of the previous variety
- Up to rotation the vectors $\begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \ldots, \begin{pmatrix} u_m \\ v_m \end{pmatrix}$ determine the tropical curve
- There is also an honest math formulation, but its omitted ☺

Thank you for your attention!

I hope that was of some help.