What is...a block?

Or: Decomposing a problem

Matrices? Well...

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{matrix} 1 & 2 & \dots & n \end{matrix} \\ \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \end{array}$$

- Over $\mathbb{C}$ we get a decomposition into matrices
  - All simple G-representations appear in $\mathbb{C}[G]$ and

    $$\mathbb{C}[G] \cong \bigoplus_{\text{simples}} L^{\oplus \dim L}$$

  - We have

    $$|G| = \sum_{\text{simples}} (\dim L)^2$$

  $\Rightarrow$ $\mathbb{C}[G] \cong \bigoplus_L M_{\dim L}(\mathbb{C})$
- In general $\mathbb{K}[G] \cong \bigoplus_{i=1}^{k} B_i$ with B_i indecomposable Blocks

Let us look at S_3

$S_3, \mathbb{C}$:

```
------------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
------------------
p  =  2   1  1  3
p  =  3   1  2  1
------------------
X.1    +   1  1  1
X.2    +   1 -1  1
X.3    +   2  0 -1
```

,

$S_3, \overline{\mathbb{F}}_2$:

```
------------------
Class |   1     3
Size  |   1     2
Order |   1     3
------------------
p  =  2   1     3
p  =  3   1     1
------------------
X.1    +   1     1
X.2    +   1 -   1
X.3    +   2    -1
```

- $\mathbb{C}[S_3] \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$ $\Rightarrow$ blocks are matrix algebras
- $\overline{\mathbb{F}}_2[S_3] \cong \overline{\mathbb{F}}_2[X]/(X^2) \oplus M_2(\overline{\mathbb{F}}_2)$ $\Rightarrow$ blocks are not necessarily matrix algebras
  (The projective cover of the trivial rep is of dim 2)

S_3 continued

$S_3, \mathbb{C}$:

```
------------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
------------------
p  =  2   1  1  3
p  =  3   1  2  1
------------------
X.1    +  1  1  1
X.2    +  1 -1  1
X.3    +  2  0 -1
```

,

$S_3, \overline{\mathbb{F}}_3$:

```
------------------
Class |   1  2
Size  |   1  3
Order |   1  2
------------------
p  =  2   1  1
p  =  3   1  2
------------------
X.1    +  1  1
X.2    +  1 -1
X.3    +  2  0 -
```

- $\mathbb{C}[S_3] \cong M_1(\mathbb{C}) \oplus M_1(\mathbb{C}) \oplus M_2(\mathbb{C})$ $\Rightarrow$ blocks are matrix algebras
- $\overline{\mathbb{F}}_3[S_3] \cong (\overline{\mathbb{F}}_3[X]/(X^3) \oplus \overline{\mathbb{F}}_3[X]/(X^3))$ $\Rightarrow$ no matrix blocks
  ($\chi_3 = \chi_1 + \chi_2$ mod 3)

For completeness: A formal statement

For $\mathbb{K}$ algebraically closed of characteristic p:

- There is an up to reordering unique decomposition

$$\mathbb{K}[G] \cong \bigoplus_{i=1}^{k} B_i \quad B_i \text{ is an indecomposable two-sided ideal summand}$$

Existence and uniqueness

- Primitive central idempotents $\overset{1:1}{\longleftrightarrow}$ blocks Construction
- The simples are partitioned by the blocks Decomposition of the problem
- Simple characters/$\mathbb{C}$ fit into blocks via simple Brauer characters

```
T := CharacterTable(SymmetricGroup(3));
Blocks(T,5);
Blocks(T,2);
Blocks(T,3);
```

```
[
    { 1 },
    { 2 },
    { 3 }
]
```

```
[
    { 1, 2 },
    { 3 }
]
```

```
[
  { 1, 2, 3 }
]
```

The principal block

```
T := CharacterTable(SymmetricGroup(10));
Blocks(T,3);
```

```
[
    { 1, 2, 5, 6, 9, 10, 11, 12, 13, 14, 19, 20, 21, 22, 29, 30, 33, 34, 35, 38,
    39, 42 },
    { 3, 8, 15, 18, 23, 26, 27, 31, 37 },
    { 4, 7, 16, 17, 24, 25, 28, 32, 36 },
    { 40 },
    { 41 }
]
```

- The principal block B_{triv} = block of the trivial rep
- B_{triv} is the "most complicated" block
- $B_{triv} \cong \mathbb{K} \Leftrightarrow \mathbb{K}[G]$ is semisimple

Thank you for your attention!

I hope that was of some help.