What are...the simples of the transformation monoid?

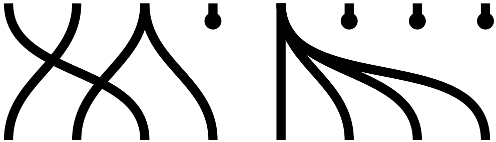
Or: Through strands, again

A reminder on H-reduction

$$\left\{\begin{matrix}\text{simples with}\\ \text{apex } \mathcal{J}(e)\end{matrix}\right\} \xleftrightarrow{\text{one-to-one}} \left\{\begin{matrix}\text{simples of (any)}\\ \mathcal{H}(e) \subset \mathcal{J}(e)\end{matrix}\right\}$$

Reps of monoids are controlled by $\mathcal{H}(e)$-cells

- We want to use this to classify simple reps for the transformation monoid T_n

(23135555) $\rightsquigarrow$  $\in T_n$

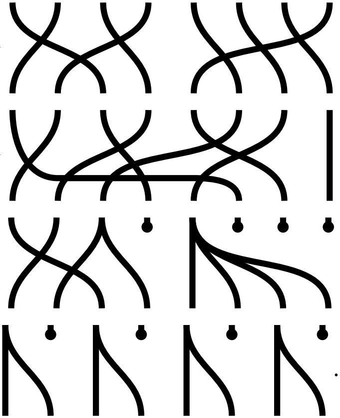
- To do list Find the J-cells, find idempotents and compute $\mathcal{H}(e)$-cells

$(24138567) \leftrightsquigarrow$

$(24637158) \leftrightsquigarrow$

$(23135555) \leftrightsquigarrow$

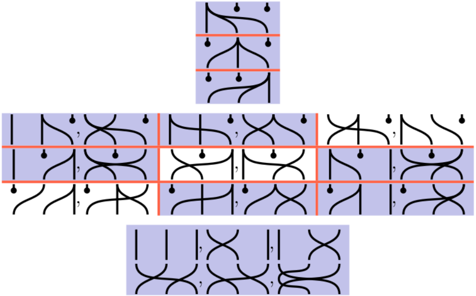
$(11335577) \leftrightsquigarrow$



- Symmetric group $S_n = \mathrm{Aut}(\{1, ..., n\})$
- Transformation monoid $T_n = \mathrm{End}(\{1, ..., n\})$
- Diagrammatic presentation using crossings, merges and dots:

crossings: , merges: , ..., dots: .

The cells - as for the Brauer monoid!

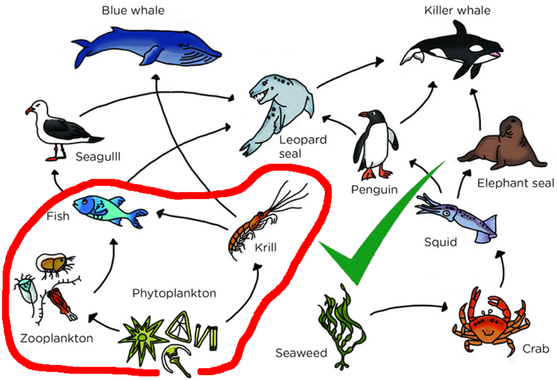


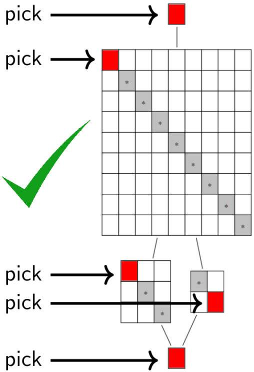
- Multiplication can only increase the number of through strands $\lambda \Rightarrow J$-cells are indexed by through strands λ
- Multiplication form the left $=$ top does not change the bottom $\Rightarrow$ left cells have fixed bottom
- Multiplication form the right $=$ bottom does not change the top $\Rightarrow$ right cells have fixed top
- $(123...(k-1)k...k)$ are idempotents for all k
- The middle is $S_\lambda \Rightarrow \mathcal{H}(e)$-cells are S_λ

For completeness: A formal statement

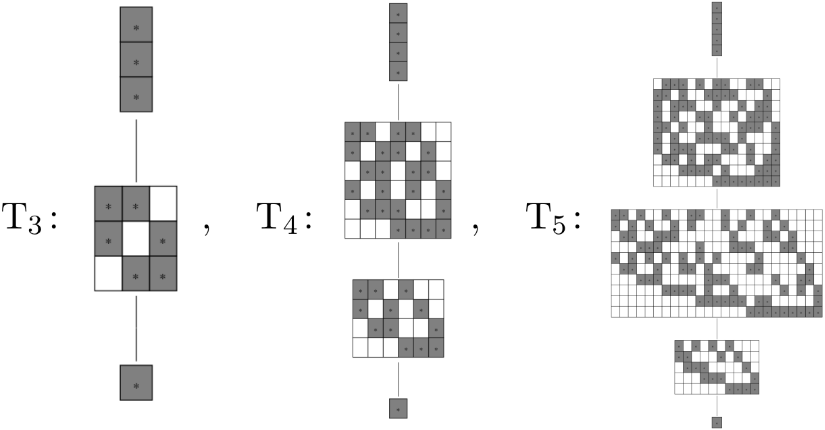
For reps of T_n over $\mathbb{C}$ we get:

- The set of apexes is $\{n, n-1, n-2, ..., 1\}$ Through strands
- There are precisely (# partitions of λ) simples of apex λ $\mathcal{H}(e) \cong S_\lambda$





We even know the simple dims!



- For K not the sign rep of S_λ the associated T_n-simple has dim $= \binom{n}{\lambda} \dim K$ with $\binom{n}{\lambda}$ = number of rows
- For K = the sign rep of S_λ the associated T_n-simple has dim $= \binom{n-1}{\lambda-1}$ Slightly unexpected

Thank you for your attention!

I hope that was of some help.