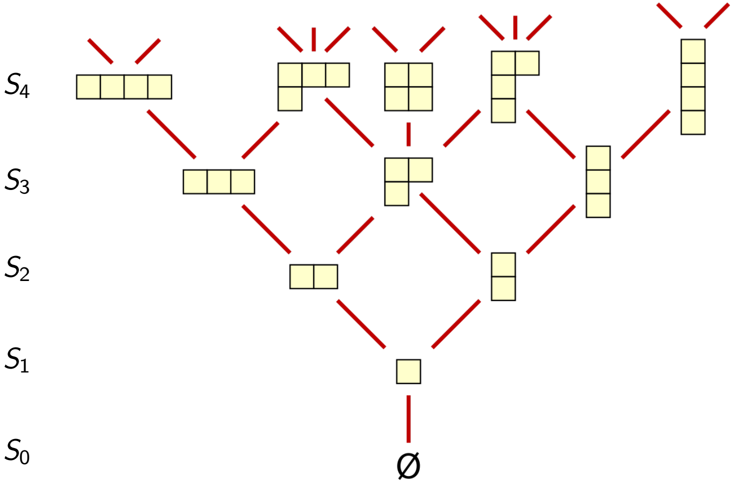
What are...Specht modules?

Or: Representations of symmetric groups, part 2

Reps of S_n



- Simple S_n reps$/\mathbb{C}$ are in bijection with Young diagrams with n boxes
- Goal Make the bijection explicit

As a vector space

$$\lambda_1 = \begin{array}{|c|c|c|}\hline & & \\ \hline \end{array}, \qquad S^{\lambda_2} = \mathbb{C}\left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \end{array} \right\}$$

$$\lambda_2 = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1} \end{array}, \qquad S^{\lambda_2} = \mathbb{C}\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} \end{array} \right\}$$

$$\lambda_3 = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array}, \qquad S^{\lambda_3} = \mathbb{C}\left\{ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \end{array} \right\}$$

- S^{λ} has a basis given by all standard Young tableaux of shape λ
- The action should be "permute numbers" but that does not quite work

“Permute numbers”

Same row: $(12).\begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 \\ \cline{1-1}\end{array} + \text{error terms}$

Same column: $(12).\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array} = -\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$

Rest: $(23).\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array} = \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}$

- Define the action only for the simple transpositions $(i, i+1)$
- Three different cases depending on i and $i+1$:
  - Same row case Eigenvalue 1 plus error terms
  - Same column case Eigenvalue -1
  - Rest Permute

For completeness: A formal statement

For all $\lambda \in \mathcal{P}(n)$ (set of partitions of n) there exists an S_n module S^λ such that:

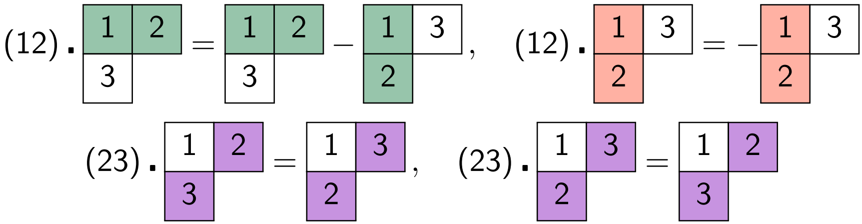
- A basis of S^λ is given by all standard tableaux of shape λ
- The action is "permute numbers" as before
- The S^λ are simple $/\mathbb{C}$
- The S^λ are pairwise nonisomorphic
- All simple S_n modules$/\mathbb{C}$ are of the form S^λ for some $\lambda \in \mathcal{P}(n)$

Die irreduziblen Darstellungen der symmetrischen Gruppe

Wilhelm Specht

Mathematische Zeitschrift **39**, 696–711 (1935) | Cite this article

Specht modules work integrally



$$(12) \leftrightsquigarrow \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix} \xrightarrow{mod\ 3} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

$$(23) \leftrightsquigarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \xrightarrow{mod\ 3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- The matrices of Specht modules have integer entries
- Specht modules can thus be defined over any field
- Catch They are in general not simple

Thank you for your attention!

I hope that was of some help.