What are...Schur's orthogonality relations?

Or: An orthonormal basis

Let us look at S_3

```
G := SymmetricGroup(3);
CT := CharacterTable(G);
CT;
```

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1    +  1  1  1
X.2    +  1 -1  1
X.3    +  2  0 -1
```

- Define an inner product by

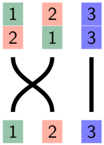
$$\langle \chi, \xi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\xi(g)}$$

- The orthogonality relations for simple characters are

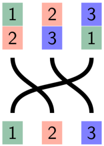
$$\langle \chi, \xi \rangle = \begin{cases} 1 & \text{if } \chi = \xi \\ 0 & \text{else} \end{cases}$$

What about non-simple reps?

S_3 acts on $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2, \mathbb{C}^2 = \mathbb{C}\{e_1, e_2\}$



e.g. $e_1 \otimes e_2 \otimes e_1 \mapsto e_2 \otimes e_1 \otimes e_1$



e.g. $e_1 \otimes e_2 \otimes e_1 \mapsto e_2 \otimes e_1 \otimes e_1$

character is ξ with

Class	1	2	3
Size	1	3	2
ξ	8	4	2

- $\langle \xi, \xi \rangle = 20$ which is not 1
- ξ is not simple

Can we decompose them?

Class	1	2	3
Size	1	3	2
ξ	8	4	2

```
-----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
-----------------
p  =  2   1  1  3
p  =  3   1  2  1
-----------------
X.1    +   1  1  1
X.2    +   1 -1  1
X.3    +   2  0 -1
```

- $\langle \xi, \chi_1 \rangle = 4$, $\langle \xi, \chi_2 \rangle = 0$, $\langle \xi, \chi_3 \rangle = 2$
- It follows that

$$\xi = 4 \cdot \chi_1 + 2 \cdot \chi_3$$

- The same decomposition then holds for the reps!

$$V_\xi \cong L_1^{\oplus 4} \oplus L_3^{\oplus 2}$$

For completeness: A formal statement

The simple characters are an orthonormal basis of all class functions

- A class function is a function $G \to \mathbb{C}$ constant on conjugacy classes
- The inner product is

$$\langle \chi, \xi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\xi(g)}$$

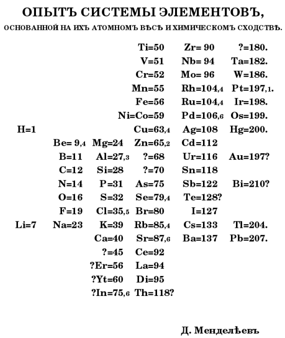
- $\langle \chi, \chi \rangle = 1 \Leftrightarrow \chi$ is simple
- # simple characters = # conjugacy classes

The orthogonality relations can aid many computations including:

- Decomposing an unknown character as a linear combination of simple characters
- Constructing the complete character table when only some of the simple characters are known
- Finding the order of the group

Constructing simple characters

```
------------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
------------------
p  =  2   1  1  3
p  =  3   1  2  1
------------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```



ОПЫТЪ СИСТЕМЫ ЭЛЕМЕНТОВЪ,

ОСНОВАННОЙ НА ИХЪ АТОМНОМЪ ВѢСѢ И ХИМИЧЕСКОМЪ СХОДСТВѢ.

			Ti=50	Zr=90	?=180.
			V=51	Nb=94	Ta=182.
			Cr=52	Mo=96	W=186.
			Mn=55	Rh=104,4	Pt=197,1.
			Fe=56	Ru=104,4	Ir=198.
			Ni=Co=59	Pd=106,6	Os=199.
H=1			Cu=63,4	Ag=108	Hg=200.
	Be=9,4	Mg=24	Zn=65,2	Cd=112	
	B=11	Al=27,3	?=68	Ur=116	Au=197?
	C=12	Si=28	?=70	Sn=118	
	N=14	P=31	As=75	Sb=122	Bi=210?
	O=16	S=32	Se=79,4	Te=128?	
	F=19	Cl=35,5	Br=80	I=127	
Li=7	Na=23	K=39	Rb=85,4	Cs=133	Tl=204.
		Ca=40	Sr=87,6	Ba=137	Pb=207.
		?=45	Ce=92		
		?Er=56	La=94		
		?Yt=60	Di=95		
		?In=75,6	Th=118?		

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- If we wouldn't know the bottom simple character, then we could construct it by solving for $a \in \mathbb{N}$, $b, c \in \mathbb{C}$:

$$1 \cdot 1 \cdot a + 3 \cdot 1 \cdot b + 2 \cdot 1 \cdot c = 0$$
$$1 \cdot 1 \cdot a + 3 \cdot (-1) \cdot b + 2 \cdot 1 \cdot c = 0$$
$$1 \cdot a \cdot a + 3 \cdot b \cdot b + 2 \cdot c \cdot c = 6$$

- Warning This is not how you want to do it in general

Thank you for your attention!

I hope that was of some help.