What is...quantum topology - part 20?

Or: Pivotal categories 2 from Chapter 4

Recall: duals

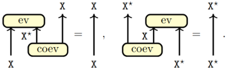
Definition 4B.1. A ***right dual*** $(X^\star, ev_X, coev_X)$ of $X \in \mathbf{C}$ in a category $\mathbf{C} \in \mathbf{MCat}$ consists of
- an object $X^\star \in \mathbf{C}$;
- a ***(right) evaluation*** ev_X and a ***(right) coevaluation*** $coev_X$, i.e. morphisms

(4B-2) $ev_X \colon XX^\star \to \mathbb{1} \rightsquigarrow$ 

, $coev_X \colon : \mathbb{1} \to (X^\star)X \rightsquigarrow$ 

;

such that they are ***non-degenerate***, i.e.

(4B-3) 

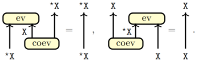
Similarly, a ***left dual*** $({}^\star X, ev^X, coev^X)$ of $X \in \mathbf{C}$ in a category $\mathbf{C} \in \mathbf{MCat}$ consists of
- an object ${}^\star X \in \mathbf{C}$;
- a ***(left) evaluation*** ev^X and a ***(left) coevaluation*** $coev^X$, i.e. morphisms

(4B-4) $ev^X \colon {}^\star XX \to \mathbb{1} \rightsquigarrow$ 

, $coev^X \colon : \mathbb{1} \to X({}^\star X) \rightsquigarrow$ 

;

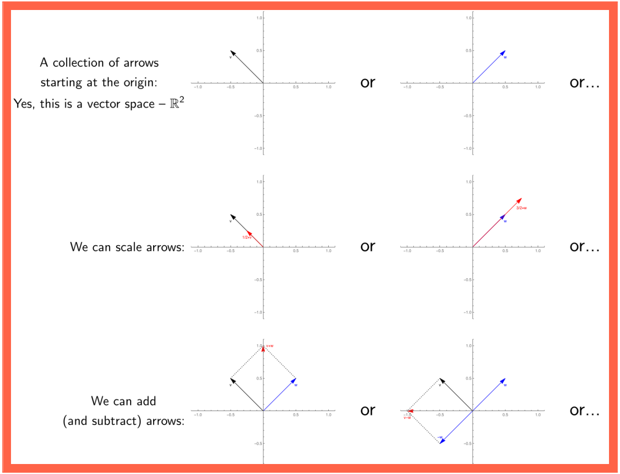
such that they are ***non-degenerate***, i.e.

(4B-5) 

We call (4B-3) and (4B-5) the ***zigzag relations***.

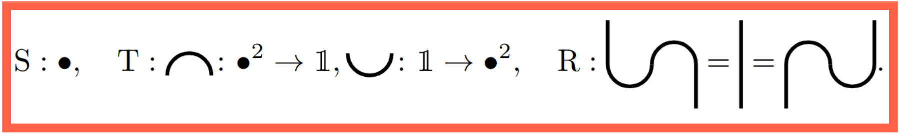
- ▶ **Above** A reminder on dual objects (think: vector space dual)
- ▶ **Rigid** = every object has a left and a right dual
- ▶ **Pivotal** "=" rigid + the double dual is itself

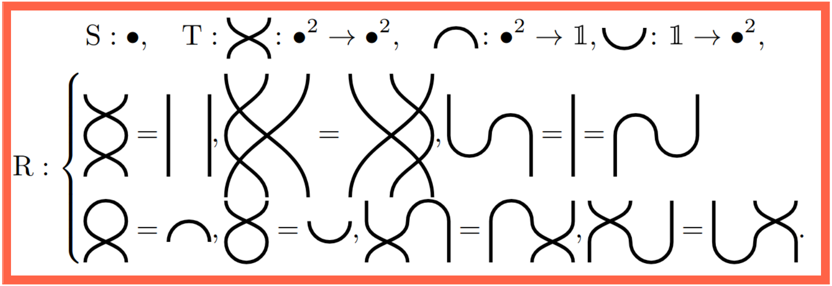
Example: vector spaces



- Example (prototypical) $\mathbf{fdVec}_{\mathbb{K}}$ is pivotal
- Double dual Here $V \cong V^*$, but only $V \cong V^{**}$ is canonical
- Careful $\mathbf{Vec}_{\mathbb{K}}$ is not rigid (only finite dimensional vector spaces have duals)

Example: diagram stuff

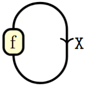


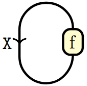


- Example (prototypical, top) **TL** is pivotal with $\bullet^* = \bullet$
- Example (prototypical, bottom) **Br** is pivotal with $\bullet^* = \bullet$
- Here cups and caps are coevaluations and evaluations

For completeness: A formal definition

Definition 4H.2. For $f \in \mathrm{End}_{\mathbf{C}}(\mathtt{X})$, where $\mathbf{C} \in \mathbf{PCat}$, we define the ***right*** $\mathrm{tr}^{\mathbf{C}}(f)$ and ***left trace*** ${}^{\mathbf{C}}\mathrm{tr}(f)$ as the endomorphisms $\mathrm{tr}^{\mathbf{C}}(f), {}^{\mathbf{C}}\mathrm{tr}(f) \in \mathrm{End}_{\mathbf{C}}(\mathbb{1})$ given by

$\mathrm{tr}^{\mathbf{C}}(f) =$ 

, ${}^{\mathbf{C}}\mathrm{tr}(f) =$ 

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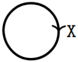
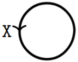
(Right trace $\rightsquigarrow$ closing $f\colon \mathtt{X} \to \mathtt{X}$ to the right, and *vice versa* for the left trace.) ◇

Definition 4H.3. For $\mathtt{X} \in \mathbf{C}$, where $\mathbf{C} \in \mathbf{PCat}$, we define the ***right*** $\dim^{\mathbf{C}}(\mathtt{X})$ and ***left dimension*** ${}^{\mathbf{C}}\dim(\mathtt{X})$ as the endomorphisms $\dim^{\mathbf{C}}(\mathtt{X}), {}^{\mathbf{C}}\dim(\mathtt{X}) \in \mathrm{End}_{\mathbf{C}}(\mathbb{1})$ given by

$\dim^{\mathbf{C}}(\mathtt{X}) = \mathrm{tr}^{\mathbf{C}}(\mathrm{id}_{\mathtt{X}}) =$ 

, ${}^{\mathbf{C}}\dim(\mathtt{X}) = {}^{\mathbf{C}}\mathrm{tr}(\mathrm{id}_{\mathtt{X}}) =$ 

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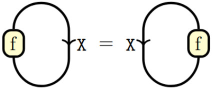
Again, the mantra is "circles = dimensions". ◇

Definition 4H.4. A category $\mathbf{C} \in \mathbf{PCat}$ is called ***spherical*** if



,

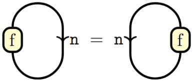
for $\mathtt{X} \in \mathbf{C}$ and all $f \in \mathrm{End}_{\mathbf{C}}(\mathtt{X})$. ◇

Example: vector spaces again

Example 4H.1. Take $\mathbf{Mat}_{\Bbbk}$, the skeleton of $\mathbf{Vec}_{\Bbbk}$, which is pivotal with

$$\mathtt{n} = \mathtt{n}^{\star} = {}^{\star}\mathtt{n}, \quad \mathrm{ev}_{\mathtt{n}} = \mathrm{ev}^{\mathtt{n}} \colon \mathtt{nn} \to \mathtt{1},\ \mathrm{ev}_{\mathtt{n}} = \begin{pmatrix} e_1 & \cdots & e_n \end{pmatrix}, \quad \mathrm{coev}_{\mathtt{n}} = \mathrm{coev}^{\mathtt{n}} \colon \mathtt{1} \to \mathtt{nn},\ \mathrm{coev}_{\mathtt{n}} = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix}.$$

Here $\{e_1, \dots, e_n\}$ denotes the standard basis of $\Bbbk^n$ (which is secretly $\mathtt{n}$, of course) . Thus, given any $\mathrm{f} = (a_{ij})_{i,j=1,\dots,n} \in \mathrm{End}_{\mathbf{Mat}_{\Bbbk}}(\mathtt{n})$, we can calculate, keeping Convention 4A.5 in mind, that

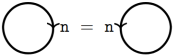


$$= \textstyle\sum_{i=1}^{n} a_{ii}.$$

This is the classical trace of the matrix f. Very explicitly, if $n = 2$ and $\mathrm{f} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then the calculation boils down to the matrix multiplication

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix}}_{\mathrm{ev}_2} \underbrace{\begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & c & d \end{pmatrix}}_{\mathrm{f} \otimes \mathrm{id}_2} \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{\mathrm{coev}_2} = a + d.$$

Moreover, we get the dimension of $\mathtt{n}$ via



$$= n.$$

Indeed, we will think of circles as dimensions. ◇

- **Note** Floating diagrams are maps from $\mathbb{K}$ to $\mathbb{K}$ a.k.a. scalars
- **Note** Traces (scalars) in $\mathbf{fdVec}_{\mathbb{K}}$ come from evaluations and coevaluations
- **Idea** Floating diagrams are traces

Thank you for your attention!

I hope that was of some help.