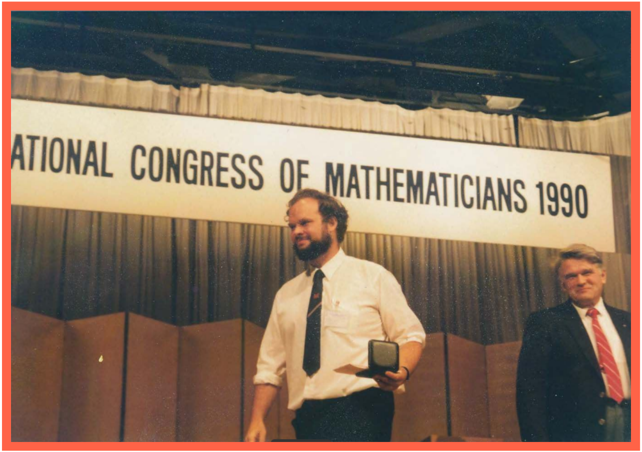
What is...quantum topology - part 2?

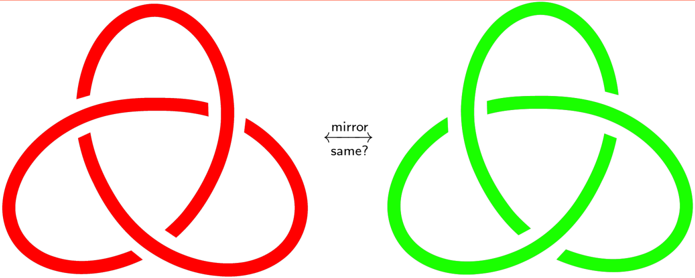
Or: The Jones polynomial

The beginning of quantum topology (QT)?



- Above Jones receives the fields medal; largely for discovering the Jones polynomial
- QT traces its roots back to the Jones polynomial
- This video explains the Jones polynomial in an intuitive, down-to-earth way

A prototypical problem



Are these the same?

Problem Building them out of rope won't help...

- Problem Find a knot invariant that can distinguish the two knots above
- From ∼1900-1980, topologists explored homology-based invariants, but these couldn't find an invariant to tell them apart
- Jones broke new ground: solving it for the first time in ∼80 years

Its everywhere!

Jones Polynomial

Jones met Birman in 1984:

Markov trace on the braid group $\Rightarrow$ knot invariant

It was surprising that the Markov trace naturally comes from the trace of the II_1 factor,

$\tau(x\sigma_n) = \tau(x)$, $\forall x \in TL_n$, $\longrightarrow$  $= \big|$, (Reidemeister move I)

therefore leading to a knot invariant, well-known as the Jones polynomial, by which Jones answered a series of old questions in knot theory in 1985.

 $= t + t^3 - t^4.$  $= t^{-1} + t^{-3} - t^{-4}.$

Reflection: $t \to t^{-1}$.

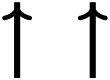
- **Above** The Jones polynomial first emerged in operator theory
- **Later** it found deep connections with many other areas of science
- That's what it's **celebrated** for: it bridges diverse fields beautifully

Enter, the theorem

There is a polynomial invariant of oriented knots and link

$$V(\_)\colon \text{knots and link} \to \mathbb{Z}[q, q^{-1}]$$

satisfying the Skein relations

$q^{-1} \cdot V\Big($  $\Big) - q \cdot V\Big($  $\Big) = (q^{1/2} - q^{-1/2}) \cdot V\Big($  $\Big)$

- V is characterized by $V(\text{unknot}) = 1$ and the Skein relations
- $V(\text{alternating})$ is an alternating polynomial
- $V(L) =_{q \leftrightarrow q^{-1}} V(\text{mirror of } L)$

  $V(\text{trefoil}) = -q^4 + q^3 + q$, so the trefoil is not equal to its mirror

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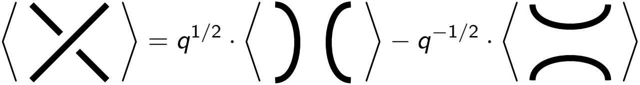
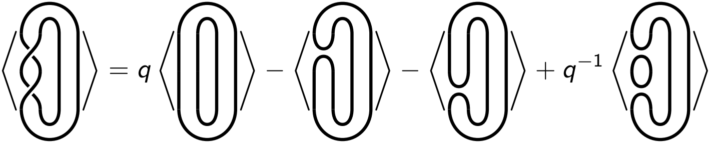
- From a worms perspective, V is powerful and easy at the same time
- From a birds perspective, V created a new field of mathematics, quantum topology, connecting various branches of modern mathematics and the sciences

On the back of an envelope – the Kauffman bracket

- $\langle \emptyset \rangle = 1$ Normalization
- $\langle \bigcirc \cup L \rangle = (q + q^{-1}) \cdot \langle L \rangle$ Pulling out circles
- Kauffman Skein



This gives the Jones polynomial up to normalization



$$= q(q+q^{-1})^2 - 2(q+q^{-1}) + q^{-1}(q+q^{-1})^2 = q^3 + q + q^{-1} + q^{-3}$$

Thank you for your attention!

I hope that was of some help.