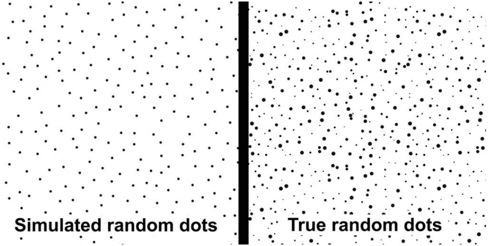
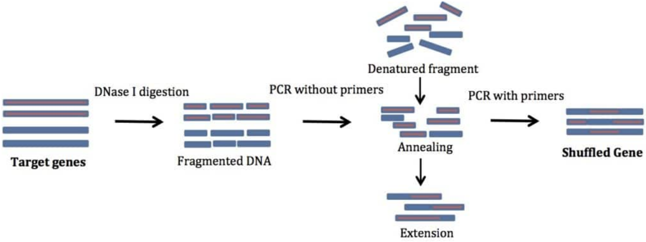
What is...card shuffling mathematically?

Or: Random walks

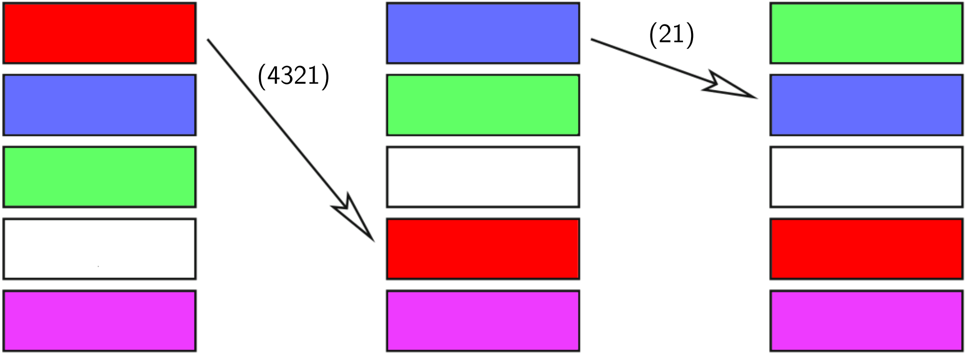
Shuffle problems are everywhere





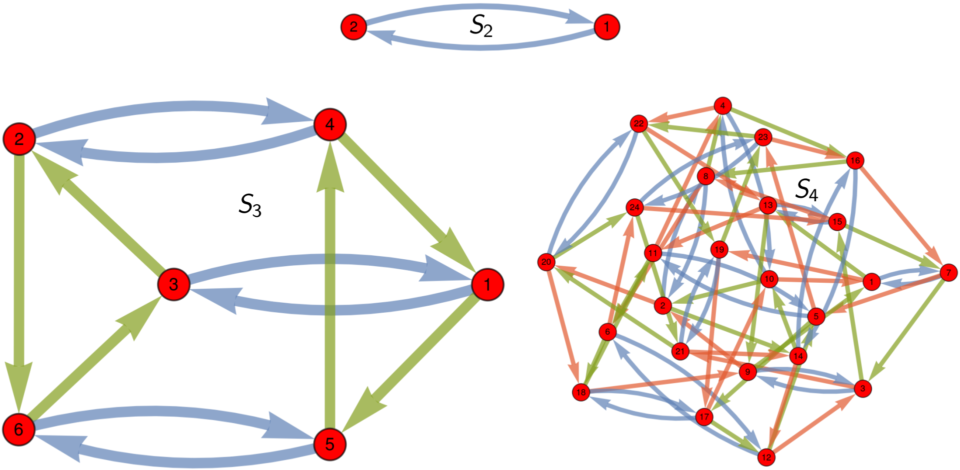
- Shuffling is everywhere: cards, diffusion, playlist, DNA, many more...
- Goal Model shuffling mathematically

An inefficient way to shuffle



- The shuffle we want to analyze is top-to-random of n cards
- Every shuffle is modeled by the permutation $(i(i{-}1)...1)$ (i to $i{-}1$ *etc.*)
- Example Above we get (4321) and (21)

Cayley graphs of permutations



- **Graph Γ** Vertexes = permutations, edges = top-to-random shuffles $(i(i-1)\ldots1)$
- **Key idea** Model top-to-random shuffles as a random walk on Γ

Enter, the theorem

Random walk on Γ (+ loops for "id shuffle") with all shuffles equally likely has:

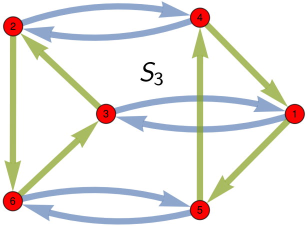
(i) The walk is ergodic **The deck will eventually be mixed**

(ii) For $c > 0$ the distance from the uniform distribution after k shuffles is

$$|\text{top-to-random}_k - \text{uniform}| \leq e^{-c}$$

where $k \geq n \log n + cn$ for n cards

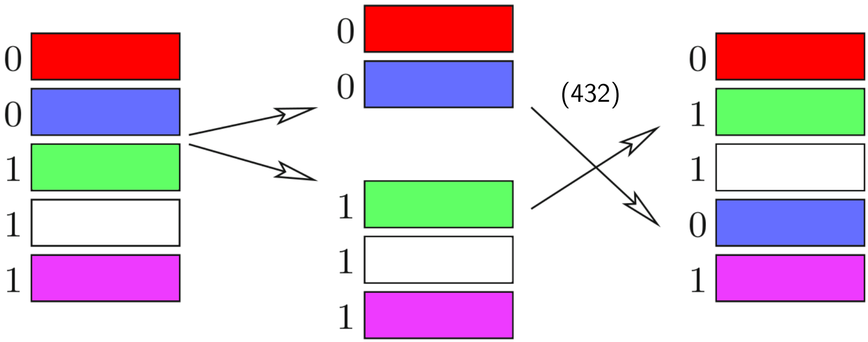
- For $n = 52$ and $c = 5$ one needs ≈ 465 shuffles **Inefficient but gets you there**
- The proof involves analyzing the spectrum of Γ via rep theory



$$\leftrightsquigarrow \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$EV : (1, \tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{3}, 0, 0)$$

Riffle shuffle random walk



- The riffle shuffle can also be modeled by a random walk
- The edges are now permutation with exactly two rising sequences
- This walk is also ergodic The deck will eventually be mixed
- One gets to the mixed state much faster than for the top-to-random shuffle

Thank you for your attention!

I hope that was of some help.