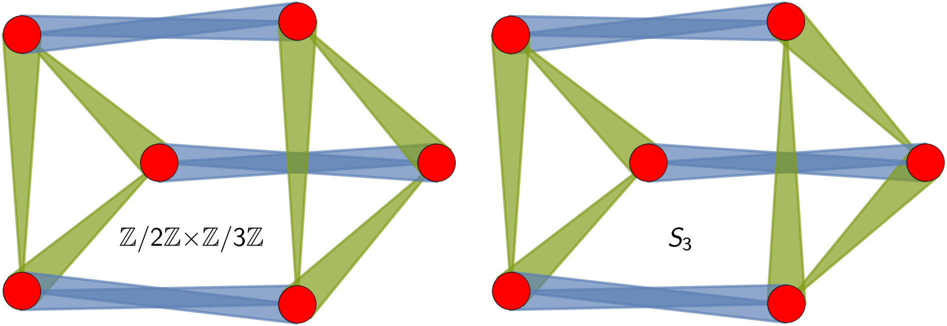
What are...spectra of Cayley graphs?

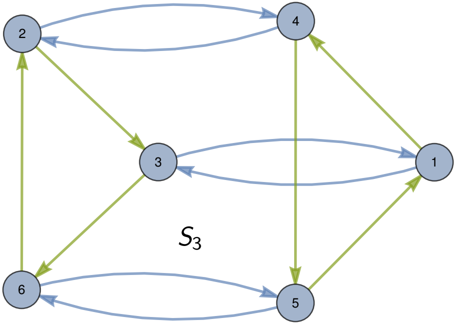
Or: Eigenvalues and characters

Graphs for group



- Cayley graphs Γ associated to group presentations $G = \langle S \rangle$
- Vertices are the group elements
- Colored edges encode the action of the generators from S
- Question What properties of G are encoded in Γ?

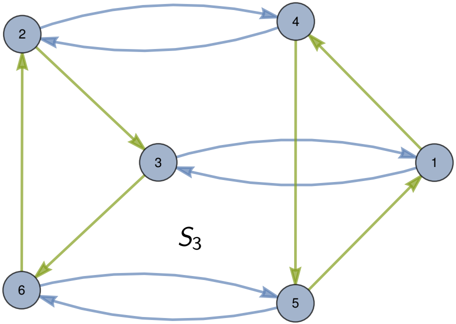
From groups to graphs to matrices



$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

- Go from a graph to a matrix via the adjacency matrix
- Matrix $\Rightarrow$ linear algebra
- Question What can linear algebra tell us about G?

Eigenvalues



$$\rightsquigarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

$$\text{Eigenvalues} : \{2, 0, 0, 0, -1, -1\}$$

- Linear algebra says: eigenvalues are useful!
- Linear algebra is trustworthy
- So we compute eigenvalues of Cayley graphs and hope for the best

Enter, the theorem

The eigenvalues of the Cayley graphs of a finite group $G = \langle S \rangle$:

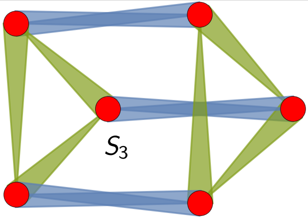
- can be indexed by the conjugacy classes of G = simple $\mathbb{C}$ reps L of G
- then appear with multiplicity $\dim L$:

$$\underbrace{EV_{L,1}, \dots, EV_{L,1}}_{\dim L}, \dots, \underbrace{EV_{L,\dim L}, \dots, EV_{L,\dim L}}_{\dim L}$$

- are given by the closed formula (χ_L = character of L)

$$EV_{L,1} + \dots + EV_{L,\dim L} = \sum_{g \in S} \chi_L(g)$$

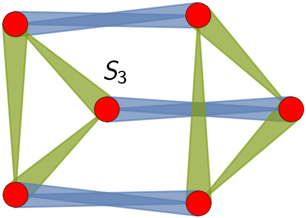
$S = \{(12), (123)\}$



$\{2, 0, 0,0,-1,-1\}$

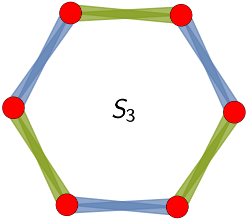
```
----------------
Class |   1  2  3
Size  |   1  3  2
Order |   1  2  3
----------------
p  =  2   1  1  3
p  =  3   1  2  1
----------------
X.1   +   1  1  1
X.2   +   1 -1  1
X.3   +   2  0 -1
```

Different Cayley graphs



$S = \{(12), (123)\}$

$\{$ 2 , 0 , 0,0,-1,-1 $\}$



$S = \{(12), (23)\}$

$\{$ 2 , -2 , 1,1,-1,-1 $\}$

Different graphs, different eigenvalues but same patterns

Thank you for your attention!

I hope that was of some help.