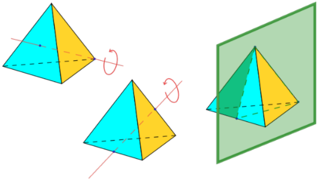
What is...Frucht's theorem?

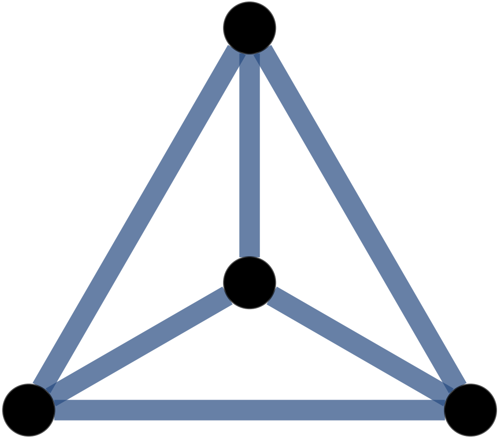
Or: Graphs and symmetries

Abstract groups and realizations

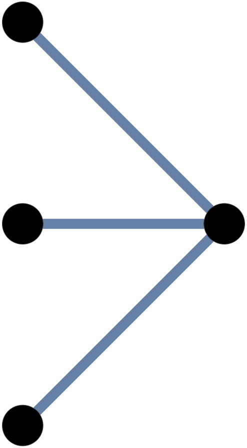
	Abstract	Incarnation
Numbers	3	 or...
Groups	$S_4 = \langle s, t, u \mid \text{some relations} \rangle$	 or...

- Groups formalize symmetry
- One group can have many real life incarnations
- Question Is 1d enough to realize groups?

Symmetries of graphs



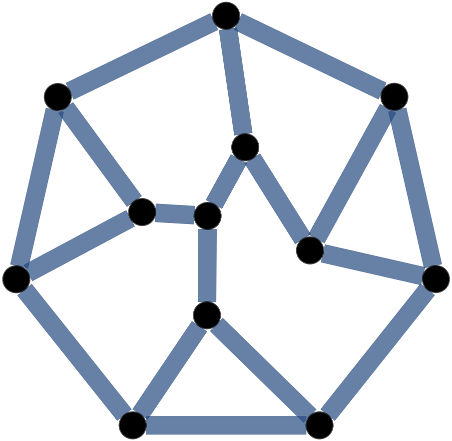
Symmetry group is S_4

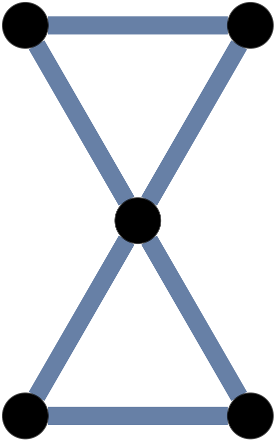


Symmetry group is S_3

- Graph automorphism = permutation of vertices keeping edge connections
- Automorphisms of a graph form a group Graph symmetry group $Sym(\Gamma)$

Here are some examples


Symmetry group is 1


Symmetry group is D_4

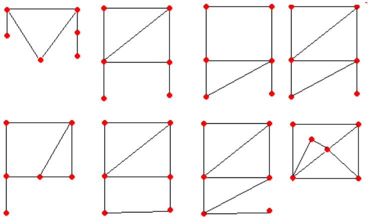
- Graphs can have very different symmetry groups
- Question Given a group G, is there a graph Γ with $Sym(\Gamma) = G$?

Enter, the theorem

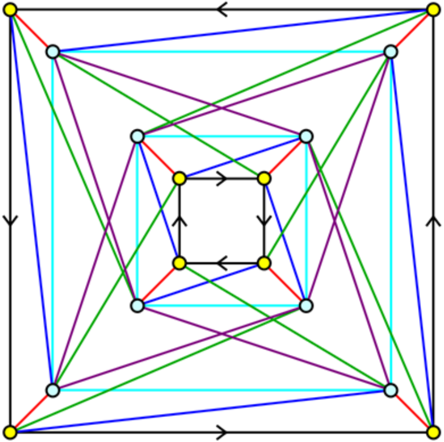
Every finite group is the group of symmetries of a finite undirected graph

- There are some stronger forms of this theorem, e.g.
  (a) One can restrict to simple graphs
  (b) There are infinitely many graphs for a given group
  (c) There are uncountably many infinite graphs realizing a given finite group
- There is even a version for infinite groups
- Some other facts are known, e.g. here is the number of asymmetric (not necessarily connected) graphs with n nodes (OEIS A003400)

1,0,0,0,0,8,152,3696,135004,7971848,805364776,144123121972



Some additional facts



Symmetry group is Q_8

- "Most" graphs have trivial automorphism group
- It is unknown whether the graph automorphism problem is P or NP-complete
- With three exceptions, one never needs more than $2|G|$ vertices

Thank you for your attention!

I hope that was of some help.