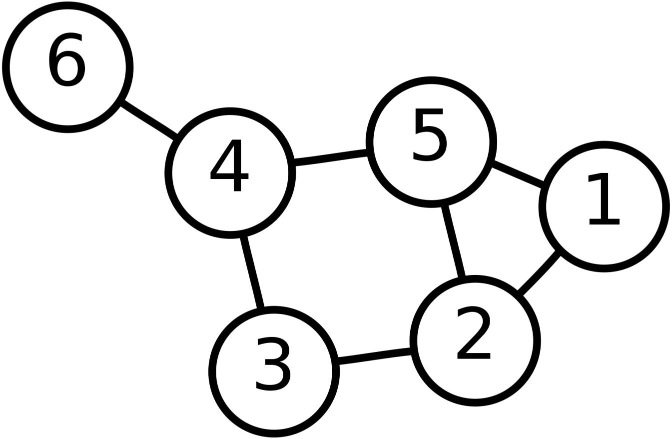
What is...the Tutte polynomial?

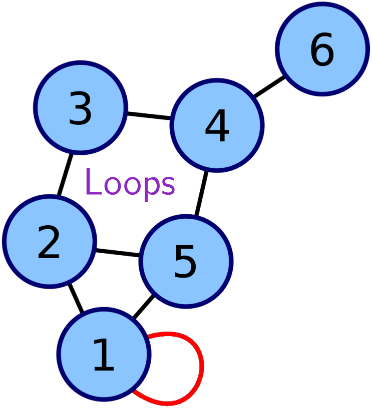
Or: Counting using polynomials

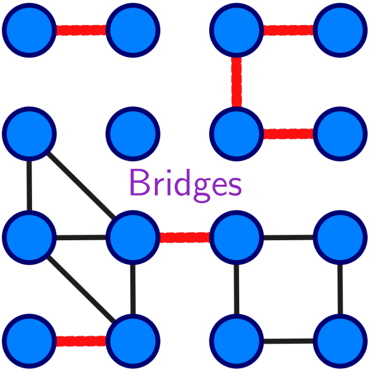
Graphs and polynomials



- Question Find a polynomial T_G that encodes properties of a given graph G
- Seems a bit unmotivated but has a very (actually many) satisfying answer
- In fact, asking this question in the first place was the brilliant idea!

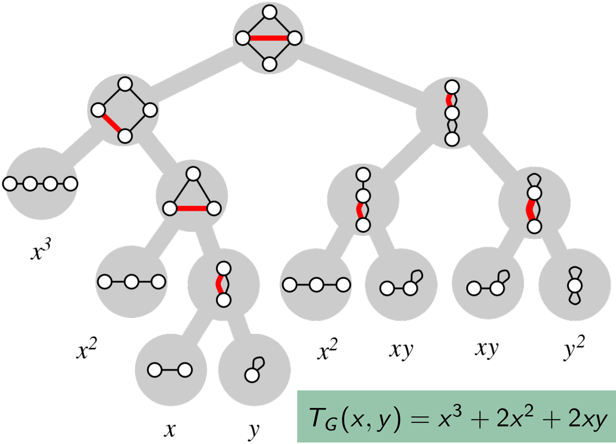
Loops and bridges





- A loop is an edge that connects a vertex to itself
- A bridge is an edge whose deletion increases the number of connected components
- These play a special role

Delete $G \backslash e$ and contract G/e



$$T_G(x,y) = x^3 + 2x^2 + 2xy + y^2 + x + y$$

For a given graph G and e neither a loop nor bridge define

$$T_G(x,y) = T_{G\backslash e}(x,y) + T_{G/e}(x,y)$$

with base case $x^m y^n$ where $m = \#\text{bridges}$, $n = \#\text{loops}$

Enter, the theorem

Deleting-contraction is well-defined and $T_G(x, y)$ has many counting properties:

(a) $T_G(2, 1)$ counts the number of forest

(b) $T_G(1, 1)$ counts the number of spanning forests

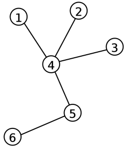
(c) $T_G(1, 2)$ counts the number of spanning subgraphs

(d) $T_G(2, 0)$ counts the number of acyclic orientations

(e) $T_G(0, 2)$ counts the number of strongly connected orientations

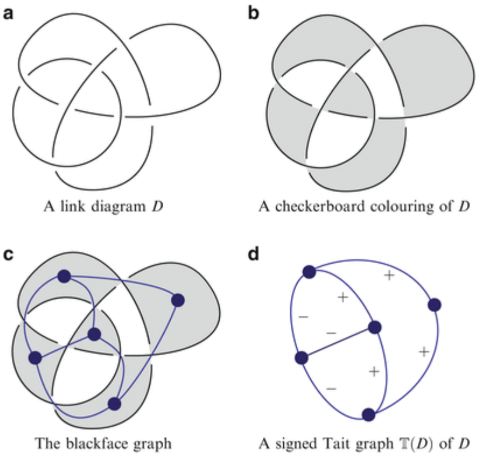
(f) More

- There is also an explicit formula
- Isomorphic graphs have the same $T_G(x, y)$ but the converse is not true



$\rightsquigarrow T_G(x, y) = x^5$ and the same for all trees with 5 edges

A knot invariant



- For an alternating knot/link take a checkerboard coloring
- This defines a graph $K(G)$
- $T_{K(G)}(x, x^{-1})$ is an invariant (the Jones polynomial of K)

Thank you for your attention!

I hope that was of some help.