What is...a tropical curve?

Or: Straight curves

Tropical semiring

	World	Addition	Multiplication	Zero	One
Classical	$\mathbb{R}$	$+$	x	0	1
Tropical	$\mathbb{R} \cup \{\infty\}$	$\oplus = \min$	$\otimes = +$	∞	0

- **Tropical addition $\oplus$** is taken min (or max)

$$4 \oplus 9 = 4, \quad 4 \oplus \infty = 4$$

- **Tropical multiplication $\otimes$** is usual addition

$$4 \otimes 9 = 13, \quad 4 \otimes 0 = 4$$

- **Tropical semiring** $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \otimes)$ is associative, commutative, distributive

$$x \otimes (y \oplus z) = (x \otimes y) \oplus (x \otimes z)$$
$$3 \otimes (7 \oplus 10) = 10$$
$$(3 \otimes 7) \oplus (3 \otimes 10) = 10$$

Tropical arithmetic

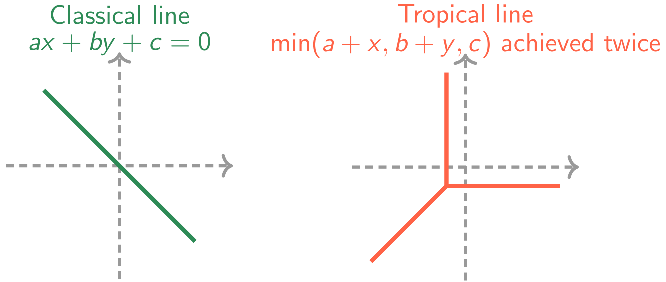
Here is a tropical *addition table* and a tropical *multiplication table*:

$\oplus$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	1	1	1	1	1	1	1
2	1	2	2	2	2	2	2
3	1	2	3	3	3	3	3
4	1	2	3	4	4	4	4
5	1	2	3	4	5	5	5
6	1	2	3	4	5	6	6
7	1	2	3	4	5	6	7

$\otimes$	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13
7	8	9	10	11	12	13	14

- Tropical arithmetic is easy
- Idea Maybe geometry over $\mathbb{T}$ is easier?
- Warning There is no subtraction! But you can divide by 0 ;-)

Tropical polynomials



- **Tropical polynomial**

$$(x \oplus y)^3 = (x \oplus y) \otimes (x \oplus y) \otimes (x \oplus y) = x^3 \oplus x^2y \oplus xy^2 \oplus y^3$$

- **Tropical Pascal's triangle**

$$\begin{array}{ccccccccc} & & & & 0 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \\ \cdots & & & & & & & & \cdots \end{array}$$

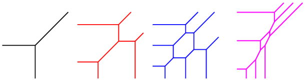
Enter, the "theorem"

The tropical vanishing set (the roots) $V(f)$ of f is

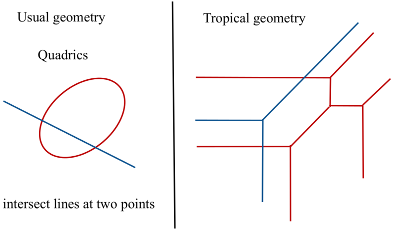
$$V(f) = \{\text{min among the terms of } f \text{ is achieved at least twice}\}$$

If f has two variables, $V(f)$ is called a tropical curve

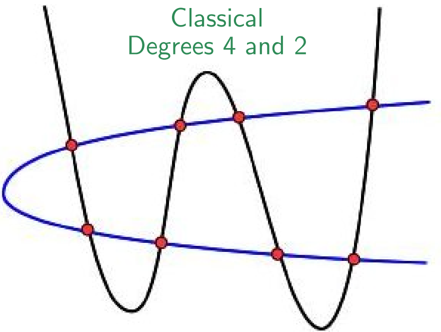
- "Theorem" Any statement in classical geometry has a nicer tropical cousin
- Tropical line, conic, cubic, *etc.*; here with max instead of min

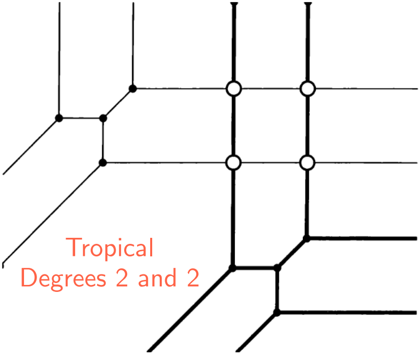


- Quadrics intersecting lines; here with max instead of min



Bézout's theorem





- Classical: Generic projective curves $/\mathbb{C}$ of degrees m, n intersect in mn points
- Tropical: Generic curves of degrees m, n intersect in mn points

Thank you for your attention!

I hope that was of some help.