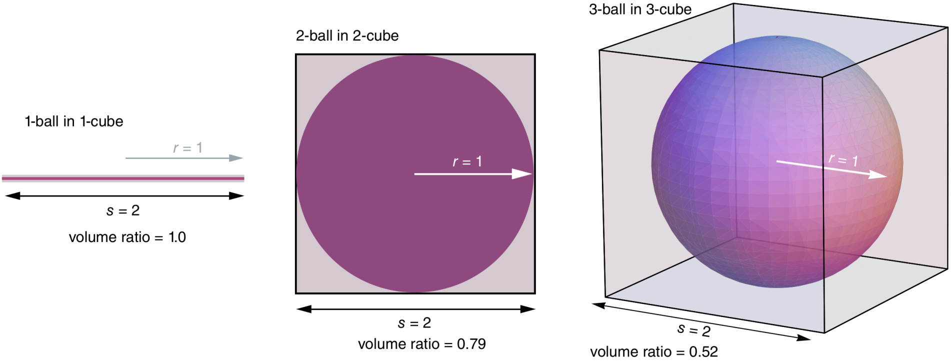
What is…the curse of dimensionality?

Or: Hyperballs do not exist! Well, kind of…

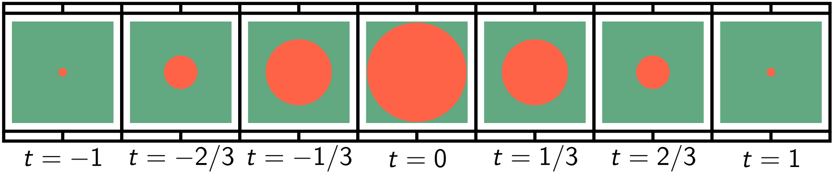
Hypercubes and hyperballs



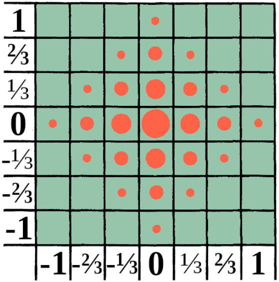
- The unit n-ball is $B^n = \{x \in \mathbb{R}^d \mid \sum_i x_i^2 \leq 1\}$ Interior of a balloon
- The surrounding n-cube is $I^n = \{x \in \mathbb{R}^d \mid -1 \leq x_i \leq 1\}$ Interior of a box
- What is the volume ratio $V(B^n)/V(I^n)$? Ball in a box

Movies!

$B^3 \subset I^3$:

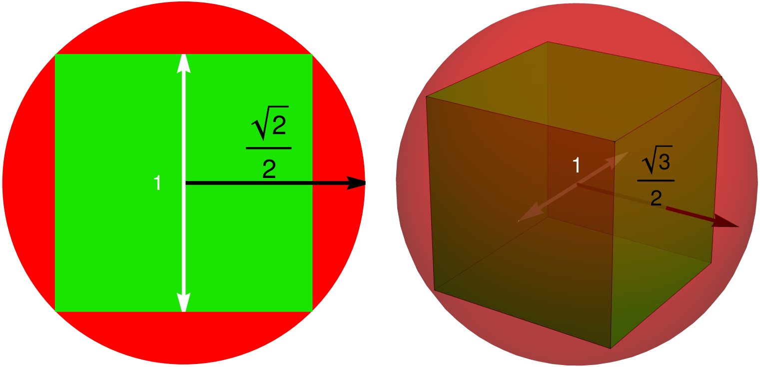


$B^4 \subset I^4$:



- Most movie frames do not contain spheres More vertices than faces
- Conjecture $V(B^n)/V(I^n) \xrightarrow{n\to\infty} 0$
- $V(I^n) = 2^n$ and we need a formula for $V(B^n)$

The surrounding n-sphere



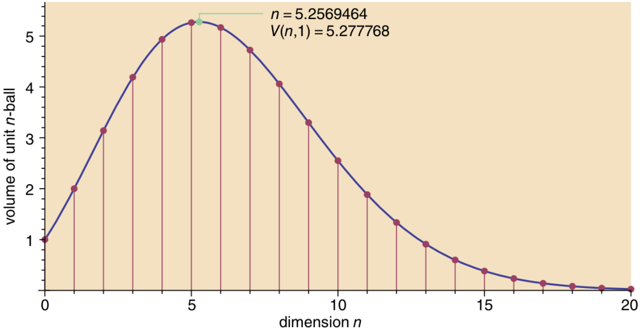
- The n-cube surrounding the unit n-ball has always side length $s = 2$
- The n-ball surrounding the unit n-cube has $r = \sqrt{n}/2 \xrightarrow{n\to\infty} \infty$
- What does this imply?

Enter, the theorem

The volume of the n-ball of radius r is

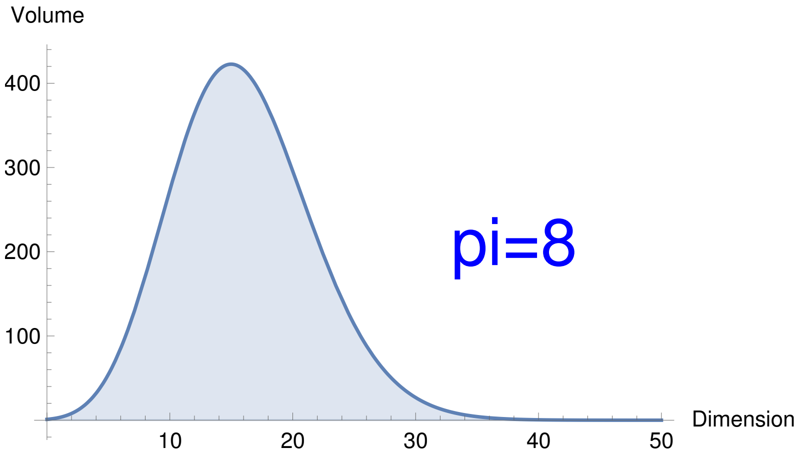
$$V(B_r^n) = \frac{1}{\Gamma(n/2+1)}\pi^{n/2}r^n = \begin{cases} \frac{1}{k!}\pi^k r^{2k} & n = 2k \\ \frac{2(k!)}{(2k+1)!}(4\pi)^k r^{2k+1} & n = 2k+1 \end{cases}$$

In particular, the volume of the unit ball goes to zero $V(B^n) \xrightarrow{n\to\infty} 0$



This is surprising because humans can not comprehend higher dimensions

Why dimension 5?



- The volume function has its peak at $n \approx 5$
- The peak is $\approx$ determined by the race between $\pi^{n/2}$ and $(n/2)!$
- So nothing is special about 5 – its about the value of π

Thank you for your attention!

I hope that was of some help.