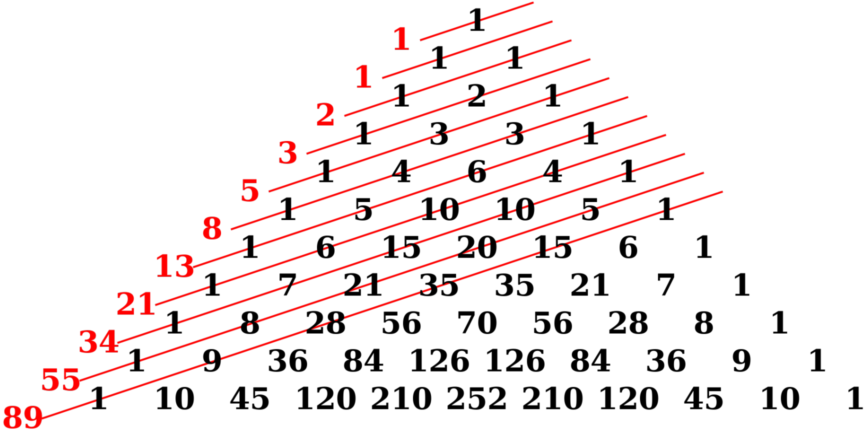
What are...Chebyshev polynomials?

Or: Fibonacci with signs?



- Fibonacci polynomials are defined by

$$F_0(X) = 0, F_1(X) = 1 \text{ and } F_n(X) = X \cdot F_{n-1}(X) + F_{n-2}(X)$$

- Coefficients $\leftrightsquigarrow$ Pascal's triangle; $F_n(1) = n$th Fibonacci number

The Chebyshev polynomials

1										
1	−1									
1	−2	1								
1	−3	3	−1							
1	−4	6	−4	1						
1	−5	10	−10	5	−1					
1	−6	15	−20	15	−6	1				
1	−7	21	−35	35	−21	7	−1			
1	−8	28	−56	70	−56	28	−8	1		
1	−9	36	−84	126	−126	84	−36	9	−1	
1	−10	45	−120	210	−252	210	−120	45	−10	1

- Chebyshev polynomials (second kind and normalized) are defined by

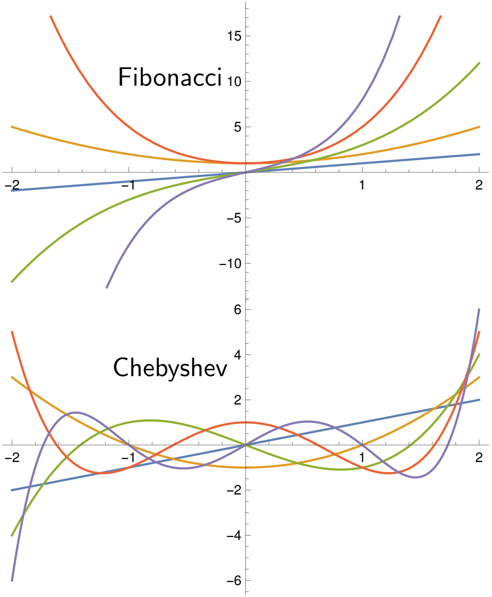
$$U_0(X) = 0, U_1(X) = 1 \text{ and } U_n(X) = X \cdot U_{n-1}(X) - U_{n-2}(X)$$

- Coefficients $\leftrightsquigarrow$ signed Pascal's triangle; $i^{n+1}U_n(i) = n$th Fibonacci number

The Chebyshev roots

Not so :

nice roots :



Nice roots :

The Chebyshev roots $2\cos(k\pi/n)$ are ubiquitous in mathematics

Enter, the theorems

Here are several facts about $U_n(X)$:

- Every algebraic integer whose conjugates are in $]-2,2[$ is a root of a $U_n(X)$; all roots of a $U_n(X)$ are algebraic integer whose conjugates are in $]-2,2[$
  "Minimal algebraic integers"
- The $U_n(X)$ for $n>0$ form a basis of $\mathbb{Z}[X]$ Division free

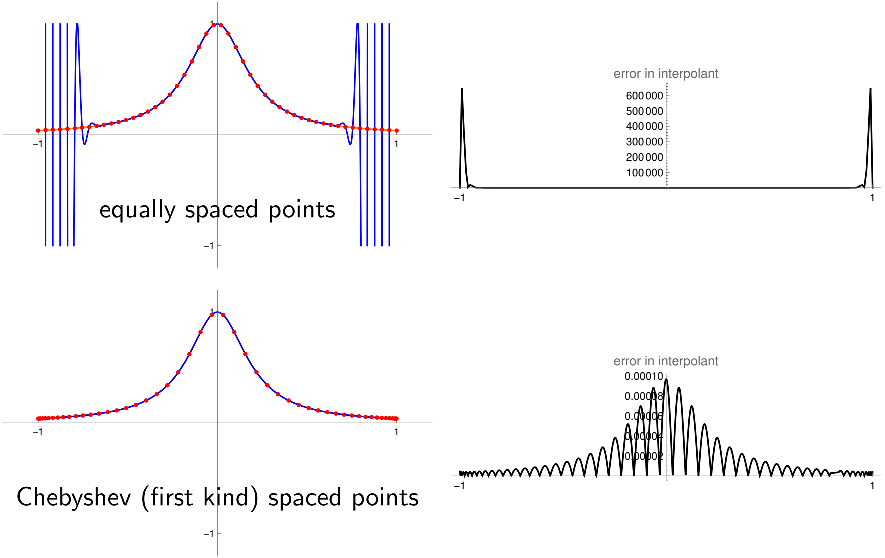
  Example. $X^5 = U_6(X) + 4 \cdot U_4(X) + 5 \cdot U_2(X)$

- The previous bullet point is even non-negative, *i.e.* $X^m = \mathbb{N}\text{-sum of } U_n(X)$
- The $U_n(X)$ are the simple characters of $\mathrm{SL}_2(\mathbb{C})$ Representation theory

$$U_1(x) = 1 \leftrightsquigarrow \mathbb{C}, \quad U_2(x) = X \leftrightsquigarrow \mathbb{C}^2,$$
$$U_3(x) = X^2 - 1 \leftrightsquigarrow \mathrm{Sym}^2\mathbb{C}^2 \subset \mathbb{C}^2 \otimes \mathbb{C}^2 \cong \mathrm{Sym}^2\mathbb{C}^2 \oplus \mathbb{C}, \quad \textit{etc.}$$

- Many more, *e.g.* classification of root and Coxeter systems, integral matrices...

Runge's phenomenon for $f(x) = 1/(1 + 25x^2)$



Error$\to \infty$ for high-degree polynomial interpolation at equidistant points

Error$\to 0$ for high-degree polynomial interpolation at Chebyshev points

Thank you for your attention!

I hope that was of some help.