What is...the Burnside problem?

Or: Obviously not! Ah, well...?

An easy question with a delicate answer

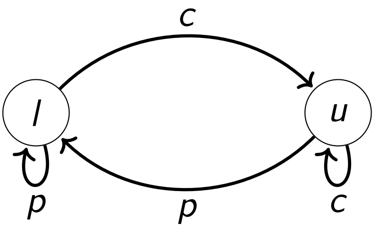
- In finite groups every element has finite order
- An element of infinite order generates $\mathbb{Z}$
- "Easy" examples of infinite groups have elements of infinite order, *e.g.*:
  - For $\mathbb{Z}$ take 1
  - For free groups take a generator
  - For $\mathrm{GL}_2(\mathbb{Z})$ take $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$
  - For braid groups take 

Burnside's question ∼1902. Is there any (finitely generated) infinite group where every element has finite order?

The answer is Yes

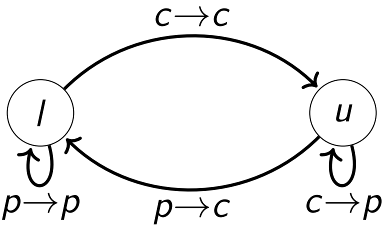
(This goes back to Golod–Shafarevich ∼1964, but I will show you a construction equivalent to one due to Grigorchuk ∼1980)

A model of a turnstile

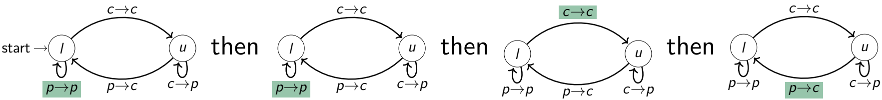


- State set $\{l = \textit{locked}, u = \textit{unlocked}\}$ States are vertices
- Alphabet $\{p = \textit{push}, c = \textit{coin}\}$ Edge labels
- Initial state $\{l\}$; this will not be important in this video
- Transition $\{p, c\} \times \{l, u\} \to \{l, u\}$ Transitions are edges

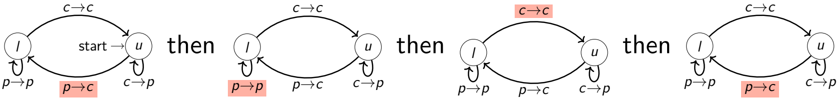
A semigroup from a turnstile



- State l is a function from c, p words to c, p words, *e.g.* $ppcp \mapsto ppcc$:



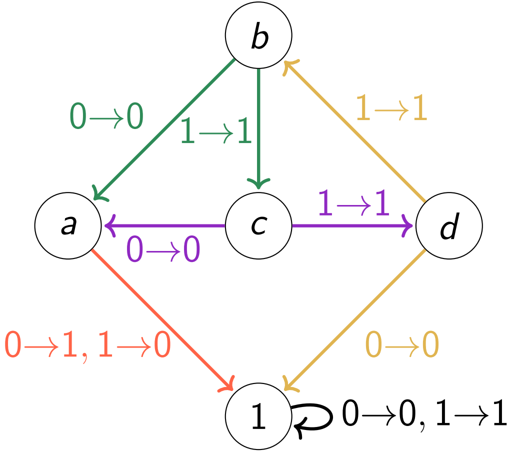
- State u is a function from c, p words to c, p words, *e.g.* $ppcp \mapsto cpcc$:



l and u are functions and generate a semigroup from the automaton turnstile

Enter, the theorem!

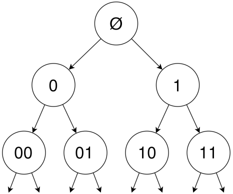
Let G be the group associated to the automaton/Mealy machine

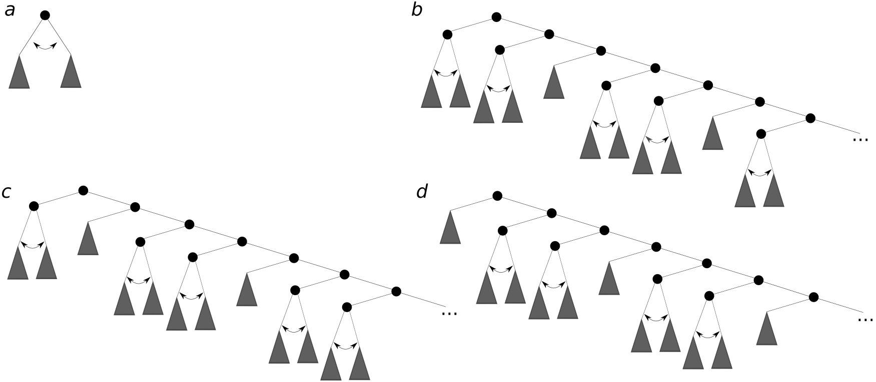


- G is Infinite
- Every element of G has order 2^k for some k Finite order
- G has many other surprising properties Link in the description

Some automorphisms of a binary tree

The group G acts on a binary tree





Thank you for your attention!

I hope that was of some help.