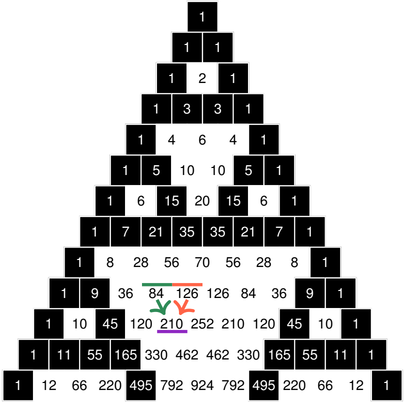
What is...the Lucas theorem?

Or: Playing with digits.

Pascal's triangle modulo a prime

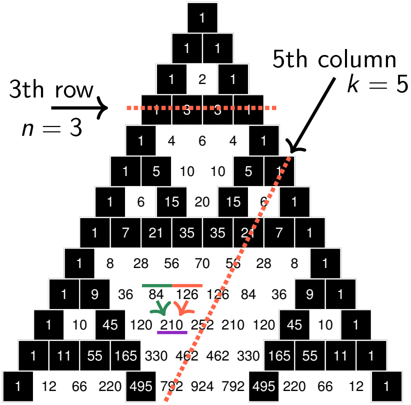
Pascal modulo 2; odd and even:



Question. Can we understand these funny patterns?

Binomials everywhere

The entries are binomials:



The trick is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$

A different number system?

$$1038 = 1 \cdot 10^3 + 0 \cdot 10^2 + 3 \cdot 10^1 + 8 \cdot 10^0 = [1, 0, 3, 8]_{10}$$

But you can try different bases, *e.g.*

$$1038 = 1 \cdot 5^4 + 3 \cdot 5^3 + 1 \cdot 5^2 + 2 \cdot 5^1 + 3 \cdot 5^0 = [1, 3, 1, 2, 3]_{5}$$

The appearing *p*-adic digits are numbers from $0, \ldots, b - 1$

Other examples include binary (base is 2) and hexadecimal (base is 16)

Enter, the theorem!

For $n = [n_r, ..., n_0]_p$ and $k = [k_r, ..., k_0]_p$ we have

$$\binom{n}{k} = \binom{n_r}{k_r} \cdot ... \cdot \binom{n_0}{k_0} \bmod p$$

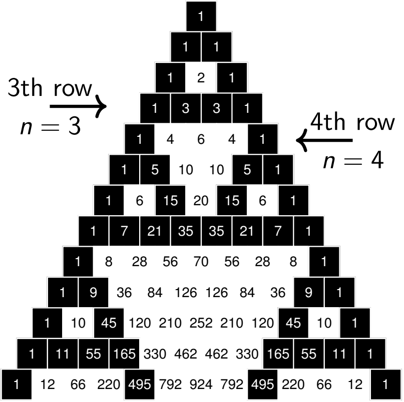
In words.

Binomials mod p are the products of the binomials along the p-adic digits

Example.

$$1038 = [6, 1, 11]_{13}, 696 = [4, 1, 7]_{13} \rightsquigarrow \binom{1038}{696} = \binom{6}{4}\binom{1}{1}\binom{11}{7} = 10 \bmod 13$$

Back to Pascal



$$3 = [1,1]_2 \rightsquigarrow \binom{3}{k} = \binom{1}{k_1}\binom{1}{k_0} = \boxed{1} \text{ mod } 2$$

$$4 = [1,0,0]_2 \rightsquigarrow \binom{4}{k} = \binom{1}{k_2}\binom{0}{k_1}\binom{0}{k_0} = \boxed{0} \text{ mod } 2 \text{ unless } k_1 = k_0 = 0$$

Thank you for your attention!

I hope that was of some help.