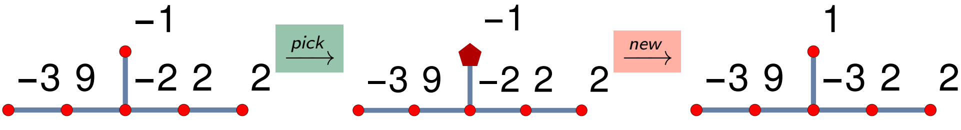
What is...the diamond lemma?

Or: Can you win?

A game on graphs

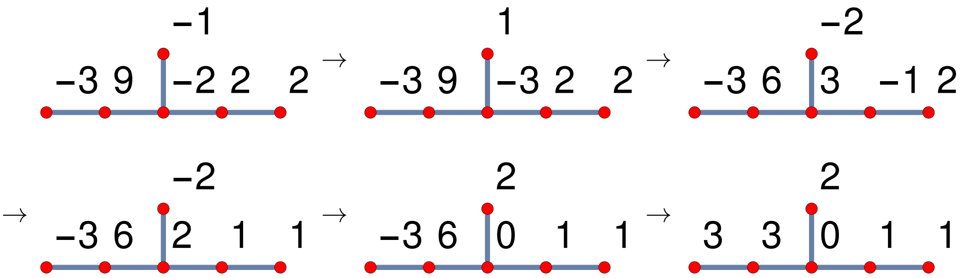
- Take an $\mathbb{R}$ vertex-weighted graph
- A move is to pick a negative vertex weight $-a$
- Get a new graph by $-a \mapsto a$ and subtracting a from the neighbors
- You win if all vertex weights are non-negative

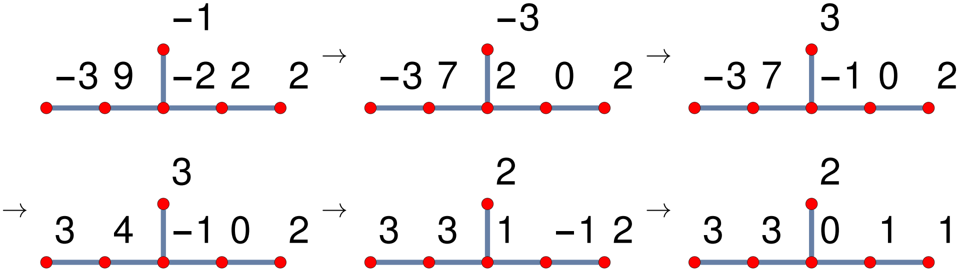


Question. If you can win, then will everyone win as well?

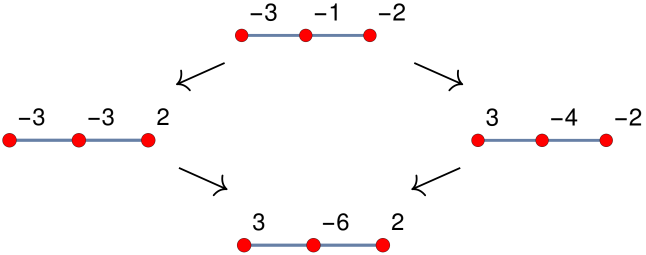
Question. Can someone do better than others?

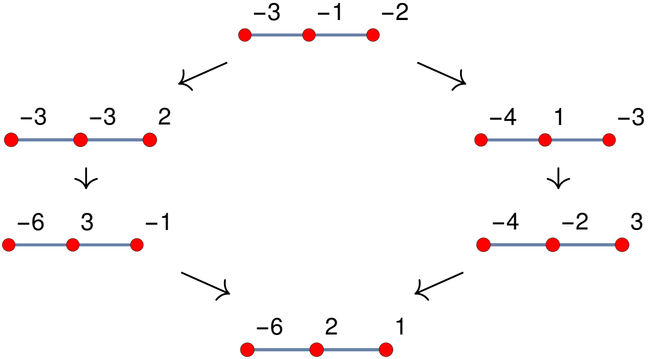
This game is unbiased – it doesn't prefer anyone?





We always have local diamonds!

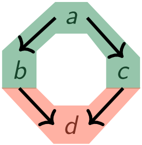




Enter, the theorem!

$\to$ binary relation on a set ($a \to b$ means that b is below a)

- Assume that there is no infinite chain $a_0 \to a_1 \to a_2 \to \ldots$
- Assume that every covering is bounded below:



Then every connected component of $\to$ as a graph contains a unique minimal element Existence and uniqueness

Widely applicable:

- PBW
- Gröbner bases
- Braid groups
- Lattices
- Noncommutative rings
- Low-dimensional topology
- Matroid theory
- More...

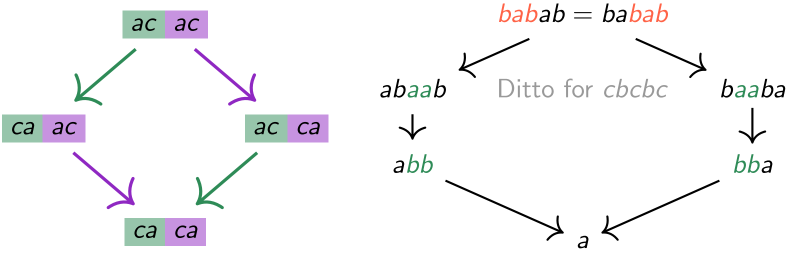
Normal forms for symmetric groups?

$$\langle a, b, c\rangle/(aa = bb = cc = 1, aba = bab, bcb = cbc, ac = ca)$$

(a) a is better than b is better than c: $bab \to aba$, $cbc \to bcb$, $ac \to ca$

(b) Shorter words are better than long words

Diamonds:



We thus always get a normal form

Thank you for your attention!

I hope that was of some help.