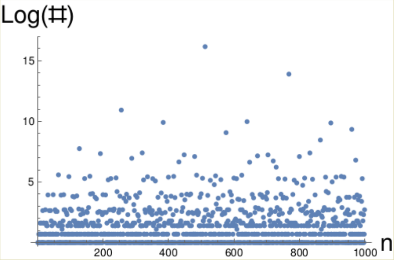
What is...counting of abelian groups?

Or: Commutative = easy

Finite groups are kind of random...

A000001 Number of groups of order n.
(Formerly M0098 N0035)

0, 1, 1, 1, 2, 1, 2, 1, 5, 2, 2, 1, 5, 1, 2, 1, 14, 1, 5, 1, 5, 2, 2, 1, 15, 2, 2, 5, 4, 1, 4, 1,
51, 1, 2, 1, 14, 1, 2, 2, 14, 1, 6, 1, 4, 2, 2, 1, 52, 2, 5, 1, 5, 1, 15, 2, 13, 2, 2, 1, 13, 1,
2, 4, 267, 1, 4, 1, 5, 1, 4, 1, 50, 1, 2, 3, 4, 1, 6, 1, 52, 15, 2, 1, 15, 1, 2, 1, 12, 1, 10, 1,

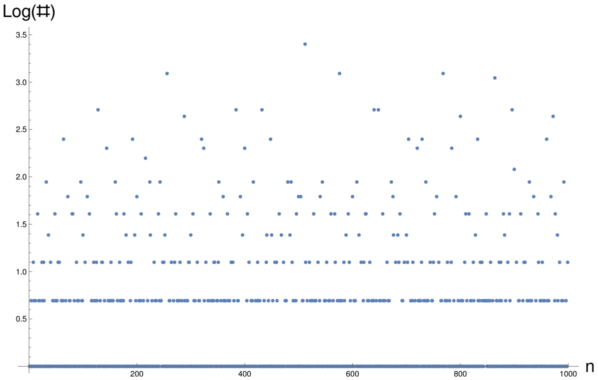


- **Groups of order n** $=$ symmetries with n operations
- **The gnu function** $gnu(n) =$ number of different groups of size n
- **Problem** We know next to nothing about $gnu(n)$

The agnu function

A000688 Number of Abelian groups of order n; number of factorizations of n into prime powers. 129
(Formerly M0064 N0020)

1, 1, 1, 2, 1, 1, 1, 3, 2, 1, 1, 2, 1, 1, 1, 5, 1, 2, 1, 2, 1, 1, 1, 3, 2, 1, 3, 2, 1, 1, 1, 7, 1, 1, 1, 4, 1, 1, 1, 3, 1, 1, 1, 2, 2, 1, 1, 5, 2, 2, 1, 2, 1, 3, 1, 3, 1, 1, 1, 2, 1, 1, 2, 11, 1, 1, 1, 2, 1, 1, 1, 6, 1, 1, 2, 2, 1, 1, 1, 5, 5, 1, 1, 2, 1, 1, 1, 3, 1, 2, 1, 2, 1, 1, 1, 7, 1, 2, 2, 4, 1, 1, 1, 3, 1, 1, 1 (list; graph; refs; listen; history; text; internal format)



- Abelian groups of order n = symmetries with n commuting operations
- The agnu function $agnu(n)$ = number of different abelian groups of size n
- Task Describe $agnu(n)$

A first answer

Chinese reminder theorem gives:

$\mathbb{Z}/1\mathbb{Z}$	1
$\mathbb{Z}/2\mathbb{Z}$	1
$\mathbb{Z}/3\mathbb{Z}$	1
$\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2
$\mathbb{Z}/5\mathbb{Z}$	1
$\mathbb{Z}/6\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, 1 \mapsto (1,1)$	1
$\mathbb{Z}/7\mathbb{Z}$	1
$\mathbb{Z}/8\mathbb{Z} \ncong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	3

- Chinese reminder theorem gives a classification of finite abelian groups
- The count comes out as follows; $n = p_1^{a_1} \dots p_k^{a_k}$ prime factorization, then:

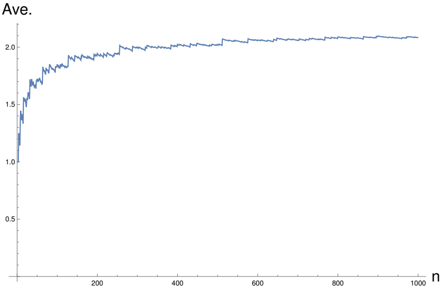
$$agnu(n) = P(a_1) \dots P(a_k) \text{ with P(a)=number of partitions of a}$$

- Example $agnu(1200) = agnu(2^4 3^1 5^2) = P(4)P(1)P(2) = 5 \cdot 1 \cdot 2 = 10$

Enter, the theorem

The average number of abelian groups of a given order is

$$\prod_{j\geq 2}\zeta(j)\approx 2.29485659$$



- For general finite groups something similar is probably out of reach
- ζ is the (Riemann) zeta function
- Average is in the sense of arithmetic mean

Semisimple rings

These have at least one ring structure (the one you get from $\mathbb{Z}/p^k\mathbb{Z}$):

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$\mathbb{Z}/1\mathbb{Z}$	1
$\mathbb{Z}/2\mathbb{Z}$	1
$\mathbb{Z}/3\mathbb{Z}$	1
$\mathbb{Z}/4\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	2
$\mathbb{Z}/5\mathbb{Z}$	1
$\mathbb{Z}/6\mathbb{Z} \xrightarrow{\cong} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}, 1 \mapsto (1,1)$	1
$\mathbb{Z}/7\mathbb{Z}$	1
$\mathbb{Z}/8\mathbb{Z} \ncong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ncong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	3

- Every abelian group has at least one ring structure
- The average number of semisimple rings is

$$\prod_{rm^2>1} \zeta(rm^2) \approx 2.49961611$$

- This number is not much bigger than the average number of abelian groups
- Semisimple = matrix rings

Thank you for your attention!

I hope that was of some help.