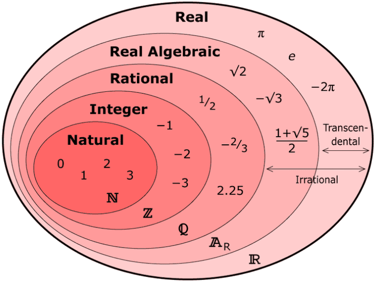
What is...the Gelfond–Schneider?

Or: Logarithms and transcendence

Transcendental numbers = beyond polynomial

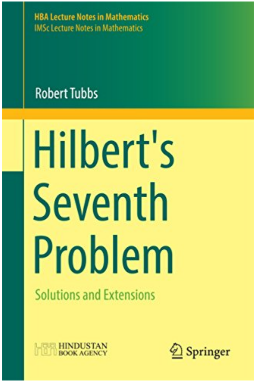


The **Gelfond–Schneider constant** or **Hilbert number**[1] is two to the power of the square root of two:

$2^{\sqrt{2}} = 2.665\,144\,142\,690\,225\,188\,650\,297\,249\,8731\ldots$

- Roots of polynomials in $\mathbb{Q}[x]$ are called algebraic
- Transcendental number = not algebraic (e.g. $2^{\sqrt{2}}$ above is transcendental)
- Proving that a number is transcendental is a classical and difficult problem
- Almost all numbers are transcendental, but proving that a specific one is transcendental is the real meat

Hilbert's seventh problem



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Are the following transcendental?

$$2^{\sqrt{2}}$$

$$e^{\pi} = (-1)^{-i}$$

$$\log 2 / \log 3$$

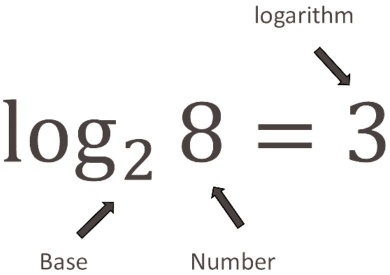
- In 1900 Hilbert gave very influential 23 problems for the 20th century
- One of them is:

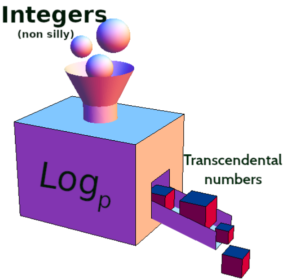
  For a, b algebraic, b irrational, is a^b transcendental?

  Equivalently,

  For a, b nonzero algebraic, $\log a / \log b$ is either rational or transcendental?

Logarithms generate transcendental numbers





- **Idea** $\log_p a$ for $a \in \mathbb{Z}_{>1}$, p prime should be transcendental if and only if $a = p^k$
- **Examples**

$$\log_2 2 = 1$$
$$\log_2 3 \approx 1.584962501$$
$$\log_2 4 = 2$$
$$\log_2 5 \approx 2.321928095$$
$$\log_2 6 \approx 2.584962501$$

Enter, the theorem

Hilbert's seventh problem is true

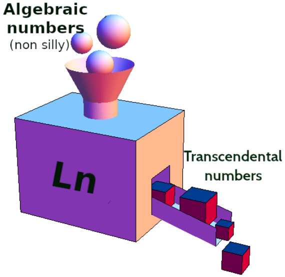
- Transcendence of many numbers remains open :

Numbers which have yet to be proven to be either transcendental or algebraic:

- Most sums, products, powers, etc. of the number π and the number e, e.g. $e\pi$, $e+\pi$, $\pi-e$, π/e, π^π, e^e, π^e, $\pi^{\sqrt{2}}$, e^{π^2} are not known to be rational, algebraically irrational or transcendental. A notable exception is $e^{\pi\sqrt{n}}$ (for any positive integer n) which has been proven transcendental.[56] It has been shown that both $e+\pi$ and π/e do not satisfy any polynomial equation of degree ≤ 8 and integer coefficients of average size 10^9.[57]
- The Euler–Mascheroni constant γ: In 2010 M. Ram Murty and N. Saradha found an infinite list of numbers containing $\frac{\gamma}{4}$ such that all but at most one of them are transcendental.[58][59] In 2012 it was shown that at least one of γ and the Euler–Gompertz constant δ is transcendental.[60]
- Apéry's constant $\zeta(3)$ (whose irrationality was proved by Apéry).
- The reciprocal Fibonacci constant and reciprocal Lucas constant[61] (both of which have been proved to be irrational).
- Catalan's constant, and the values of Dirichlet beta function at other even integers, $\beta(4)$, $\beta(6)$, ... (not even proven to be irrational).[62]
- Khinchin's constant, also not proven to be irrational.
- The Riemann zeta function at other odd positive integers, $\zeta(5)$, $\zeta(7)$, ... (not proven to be irrational).
- The Feigenbaum constants δ and α, also not proven to be irrational.
- Mills' constant and twin prime constant (also not proven to be irrational).
- The cube super-root of any natural number is either an integer or irrational (by the Gelfond–Schneider theorem).[63] However, it is still unclear if the irrational numbers in the later case are all transcendental.[citation needed]
- The second and later eigenvalues of the Gauss-Kuzmin-Wirsing operator, also not proven to be irrational.
- The Copeland–Erdős constant, formed by concatenating the decimal representations of the prime numbers.
- The relative density of regular prime numbers: in 1964, Siegel conjectured that its value is $e^{-1/2}$.
- $\Gamma(1/5)$ has not been proven to be irrational.[25]
- Various constants whose value is not known with high precision, such as the Landau's constant and the Grothendieck constant.

The Lindemann–Weierstrass theorem

	Logarithm	=	Exponent
	$\log_a N = x$	$\longleftrightarrow$	$N = a^x$
(Common Log)	$\log N = x$	$\longleftrightarrow$	$N = 10^x$
(Natural Log)	$\ln N = x$	$\longleftrightarrow$	$N = e^x$



- Transcendental number generator:

$$F(\text{non silly algebraic}) = \text{transcendental}$$

- The e^x and the natural logarithm is also a transcendental number generators
- This was known much earlier (∼1885) and shows e.g.:

$$e = e^1 \text{ is transcendental}$$

$$\pi \text{ is transcendental since } e^{\pi i} = -1 \text{ is not}$$

Thank you for your attention!

I hope that was of some help.