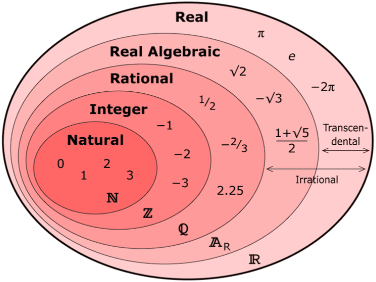
What are...Mahler functions?

Or: Self-similarity and transcendence

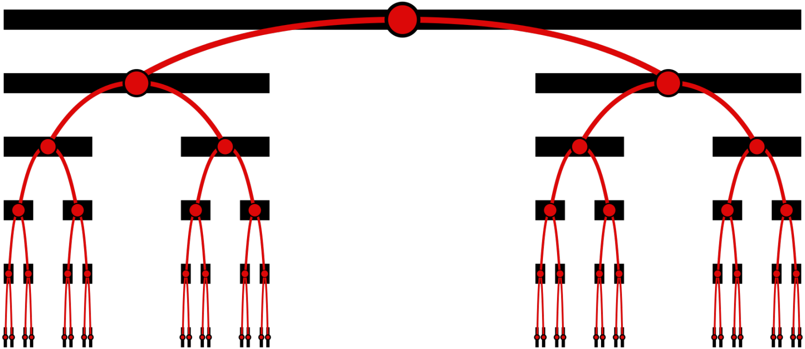
Transcendental numbers = beyond polynomial



$$L_b = \sum_{n=1}^{\infty} 10^{-n!}$$
$$= 10^{-1} + 10^{-2} + 10^{-6} + 10^{-24} + 10^{-120} + 10^{-720} + 10^{-5040} + 10^{-40320} + \ldots$$
$$= 0.\mathbf{11}000\mathbf{1}00000000000000000\mathbf{1}00000000000000000000000000000000000000000000000000$$

- Roots of polynomials in $\mathbb{Q}[x]$ are called algebraic
- Transcendental number = not algebraic (e.g. L_b above is transcendental)
- Proving that a number is transcendental is a classical and difficult problem
- Almost all numbers are transcendental, but proving that a specific one is transcendental is the real meat

Cantor sequence

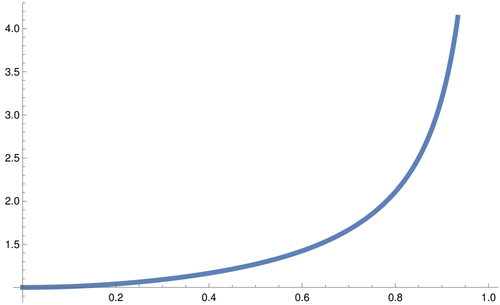


$$(\mathrm{ca}_n)_{n\in\mathbb{N}} = (1,0,1,0,0,0,1,0,1,0,0,0,0,0,0,0,0,0,1,0,1,0,0,0,1,0,1,\dots)$$

- Idea The scaled self-similarity of fractals should produce transcendental numbers
- Should (it does!) give transcendental numbers: the Cantor sequence

$$(\mathrm{ca}_n)_{n\in\mathbb{N}} \text{ with } \mathrm{ca}_n = \begin{cases} 1 & \text{if the ternary expansion of } n \text{ contains no } 1 \\ 0 & \text{otherwise} \end{cases}$$

A Mahler function

$C(w) \leftrightsquigarrow$  (pole at 1)

- Idea 2 Put fractals in generating functions
- The Cantor generating function is

$$C(w) = \sum_{n \in \mathbb{N}} \mathrm{ca}_n w^n$$

- $C(w)$ satisfies the functional equation

$$C(w) = (1 + w^2) \cdot C(w^3)$$

Enter, the theorem

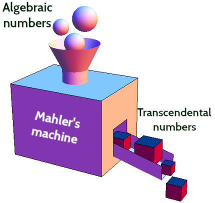
A function F: unit disc $\to \mathbb{C}$ is called s-Mahler functions of degree p whenever

$$r_0(w) \cdot F(w) = r_1(w) + r_2(w) \cdot F(w^p) + r_3(w) \cdot F(w^{p^2}) + ... + r_s(w) \cdot F(w^{p^{s-1}})$$

Here $r_i(w)$ denote rational functions, and radius of convergence is 1
Then, upon one condition (*) which is almost always true, we have:

$$F(\text{algebraic}) = \text{transcendental}$$

- Mahler functions are thus transcendental number generators



- (*) is essentially the condition that
$p(x) = r_0(1)x^s + r_2(1)x^{s-1} + ... + r_{s-1}(1)x + r_s(1)$ has no repeated roots

Generalizing Liouville's number L_b

of Mahler's theory

- **Liouville ∼1844** Number of the form

$$\sum_{k=0}^{\infty} \alpha^{k!}, \quad \alpha \text{ algebraic, } 0 < |\alpha| < 1$$

are transcendental

- **Mahler ∼1929** Number of the form

$$\sum_{k=0}^{\infty} \alpha^{p^k}, \quad \alpha \text{ algebraic, } 0 < |\alpha| < 1$$

are transcendental (the functional equation is $F(w) = w + F(w^p)$)

Thank you for your attention!

I hope that was of some help.