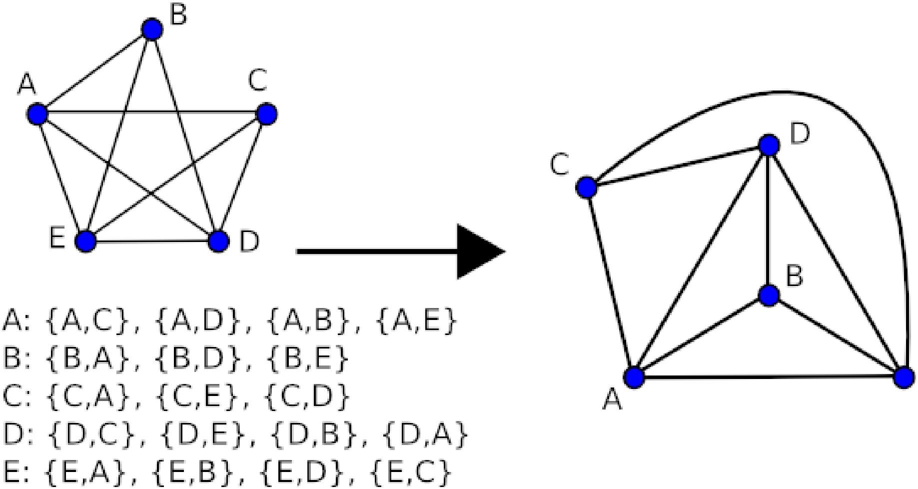
What is...planarity testing?

Or: Finding planar embeddings

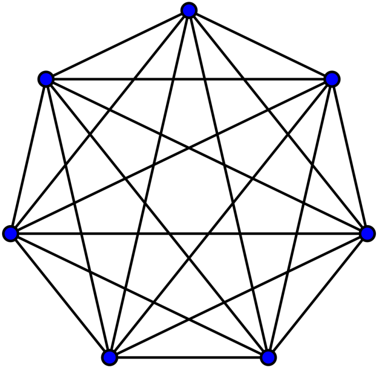
Planar graphs



- Planar graph = a graph that can be drawn in the plane without edges crossing
- Planarity is usually very tricky to decide by hand
- Question Can we effectively decide whether a graph is planar?

An easy bound

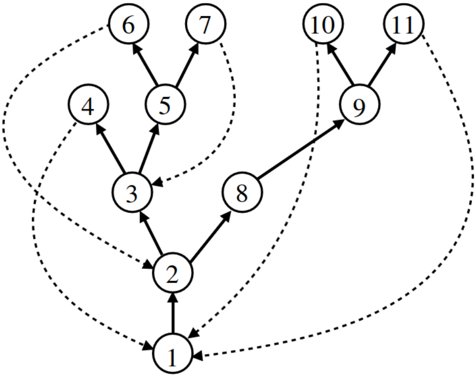
K_7:



for complete graphs K_n: $n(n-1)/2 \leq 3n-3 \Leftrightarrow n \leq 6$

- Lemma If G is planar, then #edges $E \leq$ 3#vertices V - 3 (not optimal)
- Hence, we can immediately conclude that most graphs are not planar (they have too many edges)
- The proof of this lemma is a not too difficult Euler characteristic argument

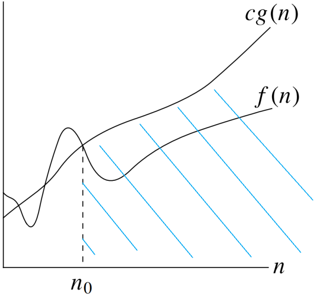
Creating palm trees



- Depth First Search (DFS) creates a palm tree out of a graph in $O(\#V + \#E)$ with backwards edges (=edges not in the tree)
- Idea Search for new neighbors until you run out of options
- Algorithm (Hopcroft–Tarjan ∼1974) Find a cycle (also $O(\#V + \#E)$) in the palm tree, delete it, check recursively planarity of the remaining pieces and the cycle, and determines whether the embeddings of these can be combined

Enter, the theorem

Hopcroft–Tarjan's algorithm runs in $O(\#V)$ and decides planarity



Order of Growth

Good

O(1)	constant time
O(log n)	logarithmic time
O(n)	linear time
O(n log n)	linearithmic
O(n^2)	quadratic time
O(n^3)	cubic time
O(2^n)	exponential time
O(n !)	factorial time

Bad

$f \in O(g)$ means f growths no faster than g

- Note that one step was $O(\#V + \#E)$ but we have $\#E \leq 3\#V - 3$
- The algorithm also finds an embedding if one exists

This planarity testing uses the Jordan curve theorem

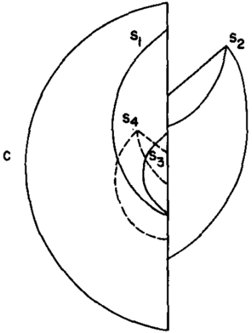


FIG. 4. Conflict between pieces. To add dotted piece S_4 on the inside of c and maintain planarity, pieces S_1 and S_3 must be moved from the inside to the outside. Piece S_2 must be moved from the outside to the inside.

- Pick a cycle c along the palm tree: this has one backwards edge; collect the remaining pieces
- Each piece can go either inside or outside of c by the Jordan curve theorem
- We add new pieces and move old pieces if necessary until either a piece cannot be added or the entire graph is embedded in the plane

Thank you for your attention!

I hope that was of some help.