What are...Borcherds–Kac–Moody algebras?

Or: Still emerging from matrices

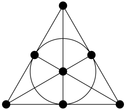
Moonshine

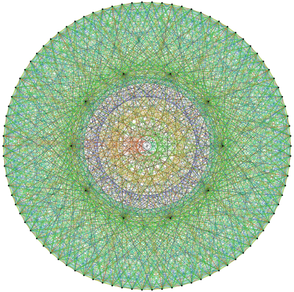
$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + 864299970q^3 + 20245856256q^4 + \cdots$$

$$\begin{aligned}
1 &= r_1 \\
196884 &= r_1 + r_2 \\
21493760 &= r_1 + r_2 + r_3 \\
864299970 &= 2r_1 + 2r_2 + r_3 + r_4 \\
20245856256 &= 3r_1 + 3r_2 + r_3 + 2r_4 + r_5 = 2r_1 + 3r_2 + 2r_3 + r_4 + r_6 \\
333202640600 &= 5r_1 + 5r_2 + 2r_3 + 3r_4 + 2r_5 + r_7 = 4r_1 + 5r_2 + 3r_3 + 2r_4 + r_5 + r_6 + r_7
\end{aligned}$$

- Two very different players: Monster sporadic group M = largest finite simple group; Klein's j-invariant j was studied as a parametrization of elliptic curves
- Moonshine observation = The coefficients of j are linear combinations of the dimensions of the simple representations of the monster group
- What is going on? Why are these different beasts related?

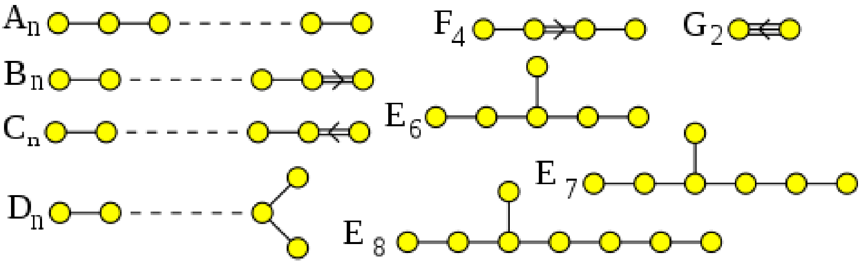
Sporadic groups are symmetries

$J_2 = Sym$ () + extra symmetry

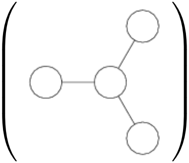
$J_2 = Sym$ ()

- The sporadic groups are symmetries of certain objects; see J_2 above
- M was first constructed as symmetries of a 196883(+1) dimension space
- Problem That does not explain the moonshine

A monster Lie algebra



$S_3 = Sym$



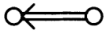
- One can construct a certain monster Lie algebra L
- $M =$ symmetries of its Dynkin-type diagram similarly to $S_3 =$ symmetries of D_4
- L has its Weyl denominator formula being $j(\sigma) - j(\tau)$

Enter, the theorem

There exists generalized Kac–Moody algebras having associated algebras

Example 2.2.4.

(1) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ 

(2) $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ 

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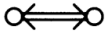
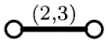
(3) $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ 

(4) $A = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$ 

but diagonal entries can be non-positive

They contain L as a special case and "moonshine follows"

- The point is that they are not just generalizations but also behave very similar (character formulas, representation theory...)
- These algebras are completely explicit and defined by generators-relations

Why are there no formulas in this video...?

So the question "What is the monster?" now has several reasonable answers:

1. It is the largest sporadic simple group or alternatively the unique simple group of its order.
2. It is the automorphism group of the Griess algebra.
3. It is the automorphism group of the monster vertex algebra. (This is probably the best answer.)
4. It is a group of diagram automorphisms of the monster Lie algebra.

Unfortunately none of these definitions is completely satisfactory. At the moment all constructions of the algebraic structures above seem artificial; they are constructed as sums of two or more apparently unrelated spaces, and it takes a lot of effort to define the algebraic structure on the sum of these spaces and to check that the monster acts on the resulting structure. It is still an open problem to find a really simple and natural construction of the monster vertex algebra.

- Most (all?) constructions of sporadic groups are rather obscure
- The construction of M using L is ok, but needs a lot of background
- There might not be any "easy and natural" construction

Thank you for your attention!

I hope that was of some help.