

What are...sporadic groups?

Or: Weird, but interesting

The periodic table

θ, C_2, Z_1 1 1	Dynkin Diagrams of Simple Lie Algebras										C_2 2						
$A_n(1), A_n(5)$ A_5 60 $A_n(3), B_n(2)$ A_6 360	$A_1(2)$ $A_1(7)$ 168 $A_1(8)$ $A_1(8)$ 504	A_n 	D_n 	E_6 	G_2 	F_4 	$E_{7,8}$ 	$B_2(3)$ $C_3(3)$ $D_4(2)$ ${}^2D_4(2^2)$ $C_6(12)$ ${}^2A_2(9)$	C_5 3	C_5 5	C_7 7	C_{11} 11	C_{13} 13	C_p p			
A_7 1520	$A_1(11)$ 660	$E_6(2)$ 25440	$E_7(2)$ 181440	$E_8(2)$ 231120	$F_4(2)$ 3315	$G_2(3)$ 428768	${}^3D_4(2^3)$ 211341312	${}^2E_6(2^2)$ 76102479481	${}^2B_2(2^3)$ 29320	${}^2F_4(2)$ 87971208	${}^2G_2(3^3)$ 104755484073	$B_5(2)$ 1451520	$C_4(3)$ 4578476	$D_5(2)$ 451489360	${}^2D_4(2^2)$ 2103337938840	${}^2A_2(25)$ 126460	C_7 7
A_8 20160	$A_1(13)$ 1102	$E_6(3)$ 137640	$E_7(3)$ 107520	$E_8(3)$ 137640	$F_4(3)$ 47544179168	$G_2(4)$ 25193680	${}^3D_4(3^3)$ 207605161912	${}^2E_6(3^2)$ 32537460	${}^2B_2(2^5)$ 26400351480	${}^2F_4(2^3)$ 58617441440	${}^2G_2(3^5)$ 439349312	$B_2(5)$ 22440720	$C_3(7)$ 441159960	$D_4(5)$ 60930008	${}^2D_4(4^2)$ 19144880	${}^2A_3(9)$ 3265920	C_{11} 11
A_9 181440	$A_1(17)$ 248	$E_6(4)$ 1196640	$E_7(4)$ 896640	$E_8(4)$ 1196640	$F_4(4)$ 1496320	$G_2(5)$ 18398008	${}^3D_4(4^3)$ 47802320	${}^2E_6(4^2)$ 64279180	${}^2B_2(2^7)$ 3449330340	${}^2F_4(2^5)$ 2291894924	${}^2G_2(3^7)$ 9323033243	$B_2(7)$ 60930008	$C_3(9)$ 9413013920	$D_3(3)$ 138207680	${}^2D_4(5^2)$ 1748026220	${}^2A_2(64)$ 5315976	C_{13} 13
A_n n	$A_1(q)$ q	$E_6(q)$ q^2	$E_7(q)$ q^3	$E_8(q)$ q^4	$F_4(q)$ q^2	$G_2(q)$ q^2	${}^3D_4(q^3)$ q^{12}	${}^2E_6(q^2)$ q^{12}	${}^2B_2(2^{n+1})$ q^{n+1}	${}^2F_4(2^{n+1})$ q^{n+1}	${}^2G_2(3^{n+1})$ q^{n+1}	$B_n(q)$ $q^{n(n+1)/2}$	$C_n(q)$ $q^{n(n+1)/2}$	$D_n(q)$ $q^{n(n-1)/2}$	${}^2D_n(q^2)$ $q^{n(n-1)}$	${}^2A_n(q^2)$ $q^{n(n-1)}$	Z_p p

- Alternating Groups
- Classical Chevalley Groups
- Chevalley Groups
- Classical Steinberg Groups
- Steinberg Groups
- Suzuki Groups
- Bor Groups and Tits Groups*
- Sporadic Groups
- Cyclic Groups

Alternates*
Order*

M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	$J(1), J(11)$	HJ	HJM	J_5	J_6	HS	McL	He	Ru
7920	90080	443520	30208960	244823040	175360	680480	50322960	6675521040	877561980	41352000	509128000	4300387308	14970034400

*The groups ${}^2F_4(q)$ and ${}^2G_2(q)$ are not a part of the Dynkin diagram. The groups ${}^2F_4(2)$ and ${}^2G_2(3)$ are the only groups that are not simple.

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*These simple groups are denominated by their order. ${}^2F_4(2)$ and ${}^2G_2(3)$ are the only groups that are not simple.

S_6	$SuzC$	$O'N$	$O'N$	C_{03}	C_{02}	C_{01}	F_4, D	HN	L_3	Ly	Th	$M(23)$	$M(23)$	$F_4, M(23)$	F_4	B	M
360	6200000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000	10000000000

- Simple group = a group without substructure (normal subgroup) = an element
- Periodic table of group theory = classification of finite simple groups; one of the main math achievements of the 20th century
- Really Everything is a group Lie-type except some sporadic examples

The Mathieu groups

M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	$J(1), J(11)$ J_1	HJ J_2	HJM J_3	J_4 86 775 371 046 077 562 880	HS	McL	$F_3, H3M, H3H$ He	Ru
7 920	95 040	443 520	10 200 960	244 823 040	175 560	604 800	50 232 960		44 352 000	898 128 000	4 030 387 200	145 926 144 000

S_2	$O'NS, O-S$	-3	-2	-1	F_5, D	LyS	F_5, E	$M(22)$	$M(23)$	$F_5, M(24)'$	F_2	F_3, M_1
Suz	$O'N$	Co_3	Co_2	Co_1	HN	Ly	Th	F_{22}	F_{23}	F_{24}'	B	M
448 345 497 600	460 815 505 920	495 766 656 000	42 305 421 312 000	4 157 776 806 543 560 000	273 030 912 000 000	51 765 179 004 000 000	90 745 943 887 872 000	64 561 751 654 400	4 089 470 473 293 004 800	1 235 205 709 190 661 721 292 800	4 154 761 000 000 000 191 177 000 000 000 000	880 037 428 794 312 075 200 000 000 000 734 707 000 754 568 000 000 000

$$M_{24} = \text{Sym} \left(\begin{array}{c} \text{Diagram of a complex polyhedral structure with 24 vertices and 192 faces, colored in shades of green, purple, and brown.} \end{array} \right) + \text{extra symmetry}$$

- ▶ The groups of Lie-type were **well-known** for a long time
- ▶ Essentially in parallel, **Mathieu ~1861** discovered the first sporadic groups
- ▶ **Construction** Augmented symmetries of naturally appearing objects

Enter, Janko

M_{11}	M_{12}	M_{22}	M_{23}	M_{24}	$J(1), J(11)$	HJ	HJM	J_4	HS	McL	H_e	Ru
7 920	95 040	443 520	10 200 960	244 823 040	175 560	604 800	50 232 960	86 775 571 046 677 562 880	44 352 000	898 128 000	4 030 387 200	145 926 144 000

S_2	$O'NS, O-S$	-3	-2	-1	F_5, D	Ly_5	F_5, E	$M(22)$	$M(23)$	$F_5, M(24)'$	F_2	F_5, M_1
Suz	$O'N$	Co_3	Co_2	Co_1	HN	Ly	Th	Fi_{22}	Fi_{23}	Fi'_{24}	B	M
448 345 497 600	460 815 505 920	495 766 656 000	42 305 421 312 000	4 157 776 806 543 360 000	273 030 912 000 000	51 745 179 004 000 000	90 745 943 887 872 000	64 561 751 654 400	4 089 470 473 293 004 800	1 235 205 709 190 661 721 292 800	4 154 761 000 226 430 191 177 580 540 000 000	880 837 428 764 512 075 800 400 900 967 748 707 000 754 548 000 000 000

2. ^ The group theorist [Bertram Huppert](#) said of J_1 : "There were a very few things that surprised me in my life... There were only the following two events that really surprised me: the discovery of the first Janko group and the fall of the [Berlin Wall](#)."

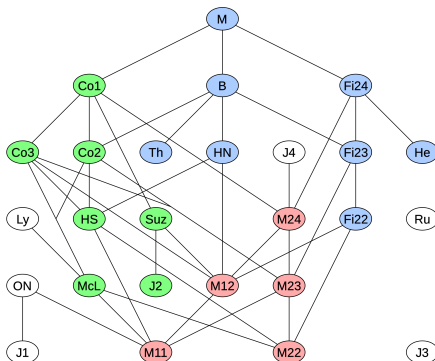
$$J_2 = \text{Sym} \left(\begin{array}{c} \text{Diagram of a tetrahedron with a circle inscribed inside, connecting vertices and midpoints of edges.} \end{array} \right) + \text{extra symmetry}$$

- ▶ **Janko ~1965** constructed the first sporadic groups since Mathieu
- ▶ **Golden era** All other sporadic groups were discovered within 20 years
- ▶ **Construction** Augmented symmetries of naturally appearing objects

Enter, the theorem

There are precisely 24 sporadic groups

- ▶ There is thus a largest sporadic group, the so-called monster
- ▶ Construction Augmented symmetries of naturally appearing objects
- ▶ Almost all sporadic groups are subgroups of the monster group M and $M =$ “symmetries of a certain monster Lie algebra”



Constructing J_2

$$J_2 = \text{Sym} \left(\begin{array}{c} \triangle \\ \triangle \\ \triangle \\ \triangle \\ \triangle \\ \triangle \\ \triangle \end{array} \right) + \text{extra symmetry}$$

$$J_2 = \text{Sym} \left(\begin{array}{c} \text{Circular graph with concentric layers of vertices and edges} \end{array} \right)$$

- ▶ J_2 is a index 2 subgroup in the automorphisms of a graph HJ on 100 vertices
- ▶ **Construction** Take the Fano plane with 7 points/lines, add 1 dummy + 21 flags (point-line pair) \Rightarrow the Fano graph on 36 vertices
- ▶ **Construction-continued** Add another dummy and 63 vertices coming from 63 involutions of $U(3,3)$

Thank you for your attention!

I hope that was of some help.