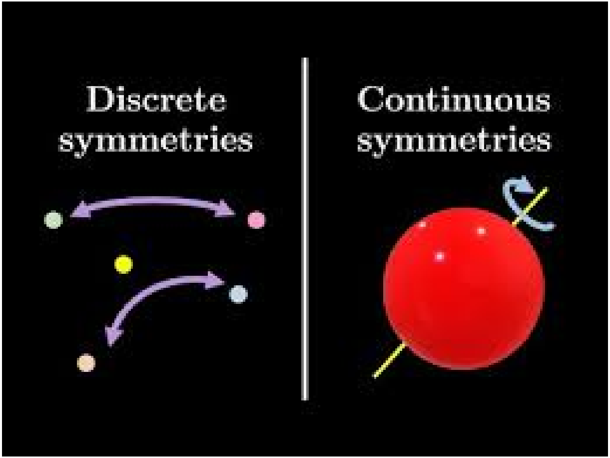
What are...Kac–Moody algebras?

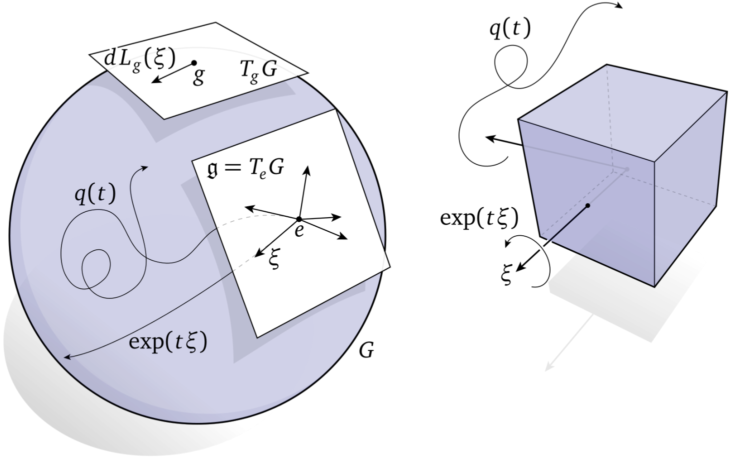
Or: Emerging from matrices

Smooth Galois theory



- Galois theory = study of symmetries of polynomial equations using finite groups
- Lie : what is the analog when studying differential equations?
- Answer Lie groups and their infinitesimal versions Lie algebras

Order one works well...!?



- These first order approximations work incredibly well
- Key formula for going between Lie groups and algebras: $\det e^A = e^{\operatorname{tr} A}$
- Example The Lie group $\mathrm{SL}_2(\mathbb{C})$ (det=1) has $\mathfrak{sl}_2(\mathbb{C})$ (tr=0) as its Lie algebra

Born from matrices

$$A_n = \begin{pmatrix} 2 & -1 & & & & & & & \\ -1 & 2 & -1 & & & & & & \\ & -1 & 2 & -1 & & & & & \\ & & -1 & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & -1 & & \\ & & & & & -1 & 2 & -1 & \\ & & & & & & -1 & 2 & -1 \\ & & & & & & & -1 & 2 \end{pmatrix} \quad n \geq 1$$

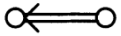
- It turns out that (nice) Lie algebras can be encoded by (Cartan) matrices
- Example Above is the matrix for $\mathfrak{sl}_{n+1}(\mathbb{C})$
- Cartan matrices certain matrices – what if we take other matrices?

Enter, the theorem

There exists generalized Cartan matrices having associated algebras

Example 2.2.4.

(1) $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ 

(2) $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$ 

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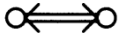
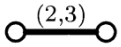
(3) $A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ 

(4) $A = \begin{pmatrix} 2 & -2 \\ -3 & 2 \end{pmatrix}$ 

These are generalizations of (universal enveloping algebras of) Lie algebras

- The point is that they are not just generalizations but also behave very similar (character formulas, representation theory...)
- These algebras are completely explicit and defined by generators-relations

Reproving number theoretical formulas

12.4. Show that for automorphisms of $s\ell_2$ of type $(s,1;1)$, where $s = 0,1,2,3$, the identity (12.3.5) turns, respectively, into the following classical identities

These are all exercises ;-)

$$\varphi(q)^3 = \sum_{n\in\mathbb{Z}}(4n+1)q^{2n^2+n} \qquad \text{(Jacobi)}$$

$$\varphi(q)^2/\varphi(q^2) = \sum_{n\in\mathbb{Z}}(-1)^n q^{n^2} \qquad \text{(Gauss)}$$

$$\varphi(q) = \sum_{n\in\mathbb{Z}}(-1)^n q^{(3n^2+n)/2} \qquad \text{(Euler)}$$

$$\varphi(q^2)^2/\varphi(q) = \sum_{n\in\mathbb{Z}} q^{2n^2+n} \qquad \text{(Gauss)}$$

- Kac–Moody algebras have an associated Weyl–Kac formula
- The formulas is completely in terms of the root system (=matrix)
- The easiest (nontrivial) version of this formula implies many well-known formulas

Thank you for your attention!

I hope that was of some help.