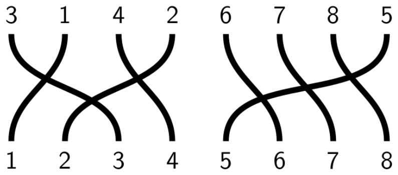
What are...Hecke algebras?

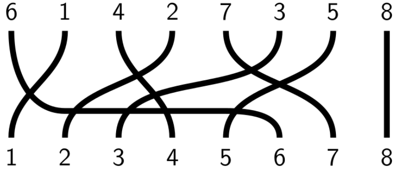
Or: Counting, or rather not counting?!

String games

Connect eight points at the bottom with eight points at the top:

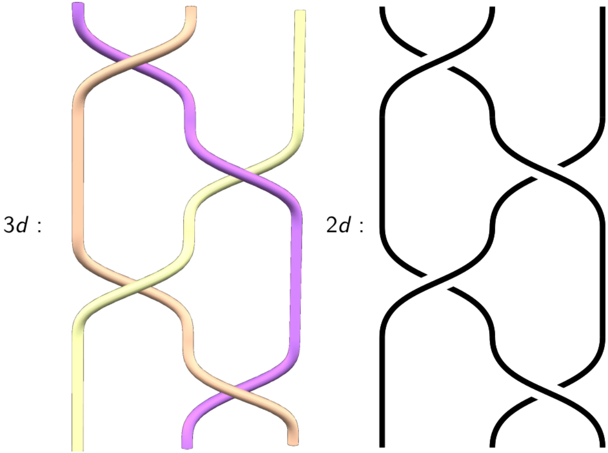


or



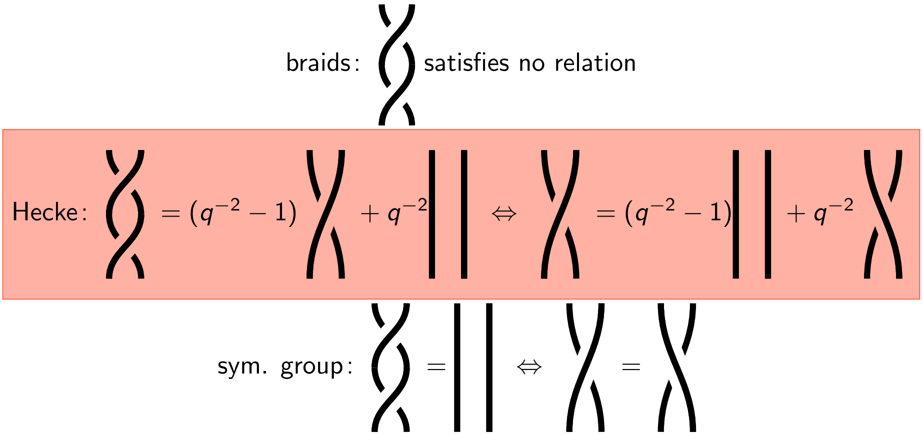
- String diagrams for symmetric groups are everywhere and beautiful ;-)
- This is a 2d version: we do not keep track of what strings goes over or under
- What is a 3d version of this picture?

Too big?



- The way to keep track over and under is the braid group
- Indeed, the symmetric group is a quotient of the braid group
- Problem The braid group is (almost always) infinite – can we do better?

Sandwich it!

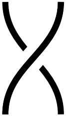


- **Idea** Impose an order two relation for the crossing
- The result should have the same size as the symmetric group but can still tell over and under apart
- And for $q = 1$ everything collapses to the symmetric group

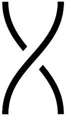
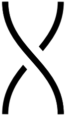
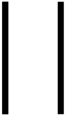
Enter, the theorem

The Hecke algebra H_n for the symmetric group S_n on $\{1, \dots, n\}$ is, by definition:

(i) The $\mathbb{Z}[q, q^{-1}]$-algebra generated by $T_1, \dots, T_{n-1}$:

$T_i \leftrightsquigarrow$  (strand i and $i+1$)

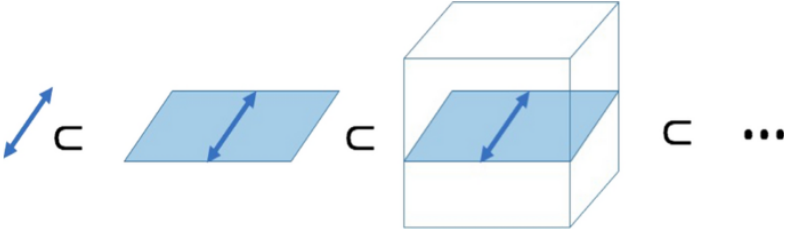
(ii) Subject to the generating relations; everything braids satisfy and:

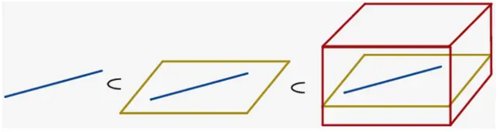
 $= (q^{-2} - 1)$  $+ q^{-2}$ 

H_n is $\mathbb{Z}[q, q^{-1}]$-free of rank $|S_n|$

- H_n is a quotient of the (group ring of the) braid group
- S_n is a specialization of H_n
- This actually works for any Coxeter group (and beyond)

Counting points...kind of...





- $G_q = \mathrm{GL}_n(\mathbb{F}_q)$ and $B_q \subset G_q$ = upper triangular matrices
- Take the flag manifold G_q/B_q
- Theorem The Hecke algebra (over $\mathbb{F}_q$ with '$q = q$') is

$$H_n \cong \mathrm{End}_{\mathbb{F}_q G_q}(\mathbb{F}_q G_q/B_q)$$

Thank you for your attention!

I hope that was of some help.