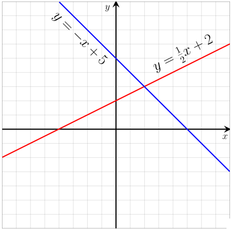
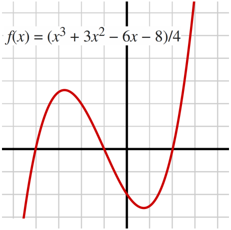
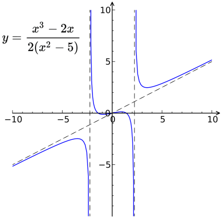
What is...Padé approximation?

Or: Better than Taylor

Approximation using easier functions

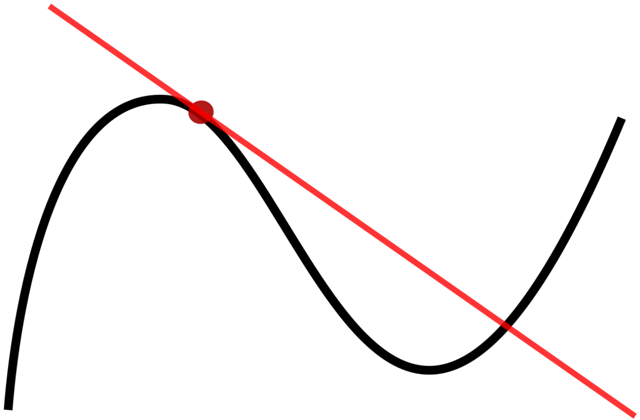
Linear: 

, Polynomial: 

Rational: 

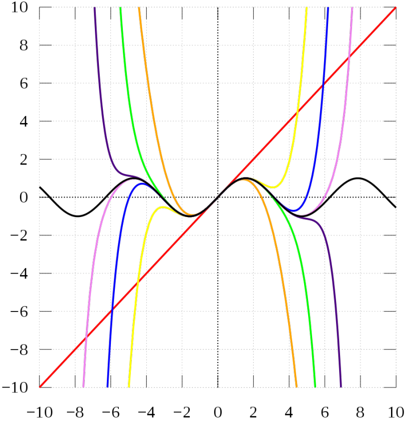
- Analysis is powerful but difficult
- Algebra is less powerful but much easier
- Idea Approximate (nice) functions of analysis using functions of algebra

Step 1: linear approximation



- The tangent is a linear approximation for a function
- As an approximation this is not all that great
- First idea Take higher derivatives into account

Step 2: polynomial approximation

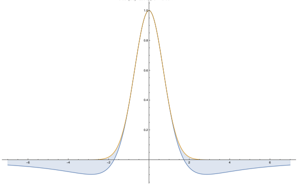


- Cut-offs of the Taylor expansion are polynomial approximations for a function
- As an approximation this is ok but still not perfect
- Idea Why not use rational functions?

Enter, the theorem

There exists a "best" approximation of a function near a specific point by a rational function $R(x)$ of given order

$$R(x) = \frac{\sum_{j=0}^{m} a_j x^j}{1 + \sum_{k=1}^{n} b_k x^k} = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}$$

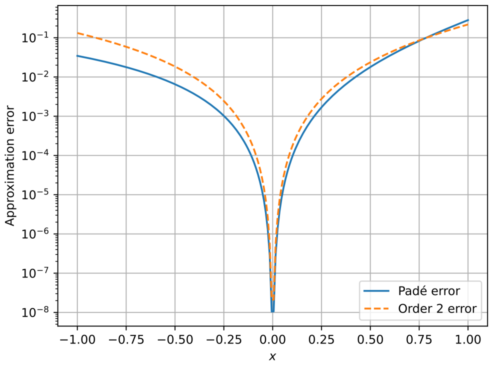


- This Padé approximation can be computed from the Taylor series
- "Best" means:

$$\begin{aligned} f(0) &= R(0), \\ f'(0) &= R'(0), \\ f''(0) &= R''(0), \\ &\vdots \\ f^{(m+n)}(0) &= R^{(m+n)}(0). \end{aligned}$$

The approximation is quite good

$$\exp(x) \approx 1 + x + 1/2 \cdot x^2 \quad \text{versus} \quad \exp(x) \approx (2+x)/(2-x)$$



- The Padé approximation is often better than Taylor even when using lower orders
- It also may still work where the Taylor series does not converge
- Padé's approximation improves the method truncating a Taylor series

Thank you for your attention!

I hope that was of some help.