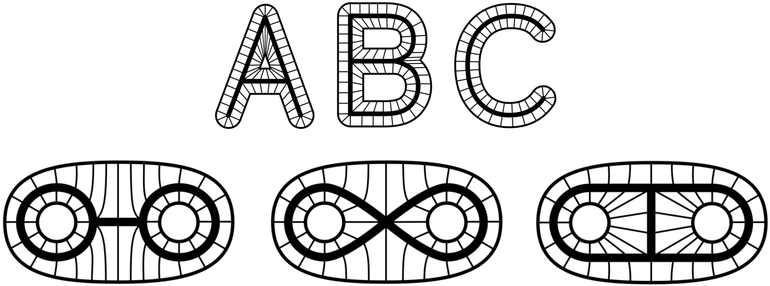
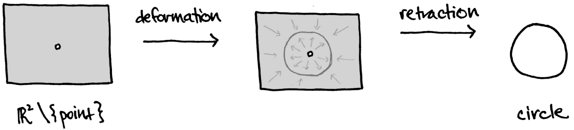
What is...homotopy type theory?

Or: HoTT = Ho + TT

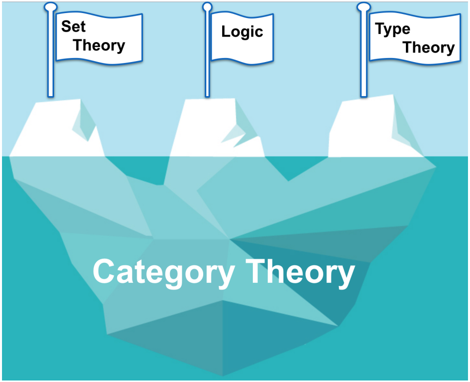
Homotopy (Ho)





- Ho is the study of spaces up to continuous deformation called homotopies
- Homotopies are main concepts of algebraic topology since homotopic spaces have the same algebraic invariants
- It took a while to take off due to its slightly strange behavior for geometric topology

Type theory (TT)



TT is maybe more category theory than set theory

- TT was invented to avoid paradoxes in set theory using the notion of types
- Types are main concepts of computer science and they motivated data types
- It took a while to take off due to its slightly complicated formulation for set theory

Homotopy type theory (HoTT)

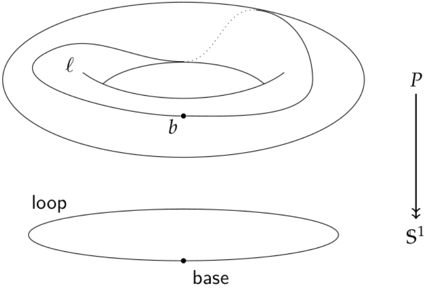


Figure 6.1: The topological induction principle for $\mathbb{S}^1$

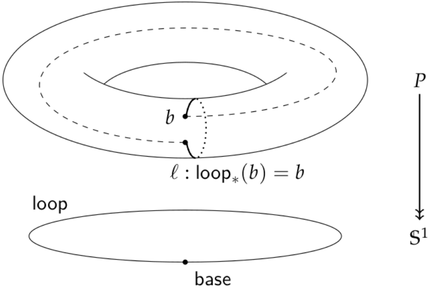


Figure 6.2: The type-theoretic induction principle for $\mathbb{S}^1$

Logic mimics topology

- HoTT interprets type theory using homotopy
- Idea Types are spaces, called homotopy types, and logical constructions are homotopy-invariant constructions on spaces
- First upshot Manipulate spaces without having to develop point-set topology or $\mathbb{R}$

Enter, the theorem

In HoTT one can do both, define the sphere S^n as a type, and define homotopy groups $\pi_n(A)$ of types and one gets the famous

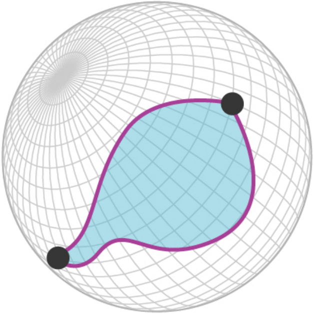
$$\pi_n(S^n) \cong \mathbb{Z}$$

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8	π_9	π_{10}	π_{11}	π_{12}	π_{13}	π_{14}
S^0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2 \times \mathbb{Z}/84$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z} \times \mathbb{Z}/12$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/3 \times \mathbb{Z}/24$	$\mathbb{Z}/15$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2 \times \mathbb{Z}/12 \times \mathbb{Z}/120$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/30$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^3$
S^6	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/60$	$\mathbb{Z}/2 \times \mathbb{Z}/24$
S^7	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$	$\mathbb{Z}/120$
S^8	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0	$\mathbb{Z}/2$
S^9	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0	0
S^{10}	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$	0
S^{11}	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/24$
S^{12}	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
S^{13}	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}/2$
S^{14}	0	0	0	0	0	0	0	0	0	0	0	0	0	$\mathbb{Z}$

This is a new method of proof in classical Ho

- The (first) proof uses a crucial axiom only accessible in HoTT
- This proof can be formalized in a proof assistant and verified by a computer
- In fact, one main application is in proof verification and automated proof writing

No equality, please



Path Equivalence

Two paths are homotopy equivalent

if there is at least one surface linking one to the other.

- HoTT replaces equality by homotopy
- Equality $a = b$ becomes a path $a \to b$ in a space
- This makes topology appear in logic – "surfaces between equal terms"

Thank you for your attention!

I hope that was of some help.