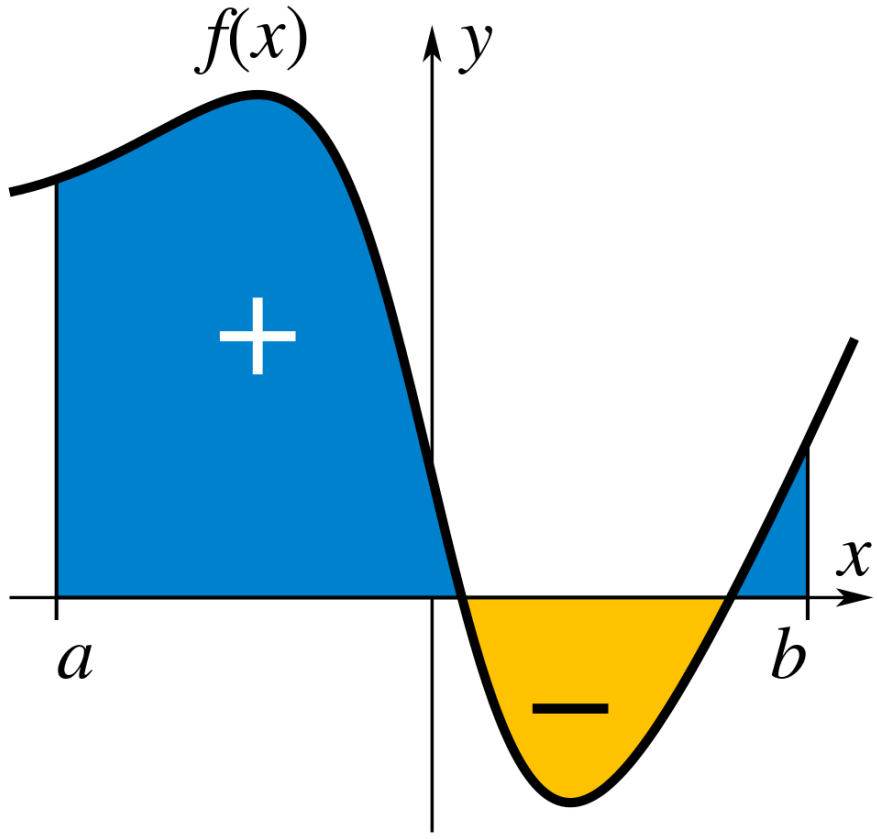
What is...symbolic integration?

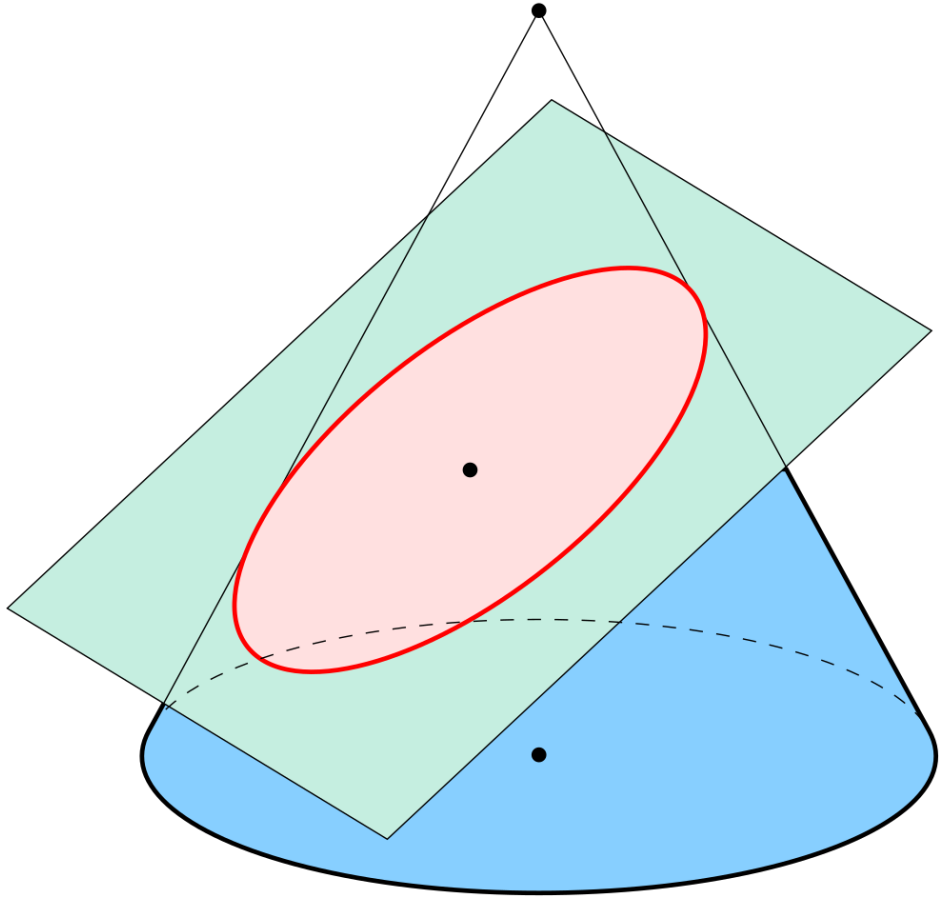
Or: Symbolic integration rules!

Integration



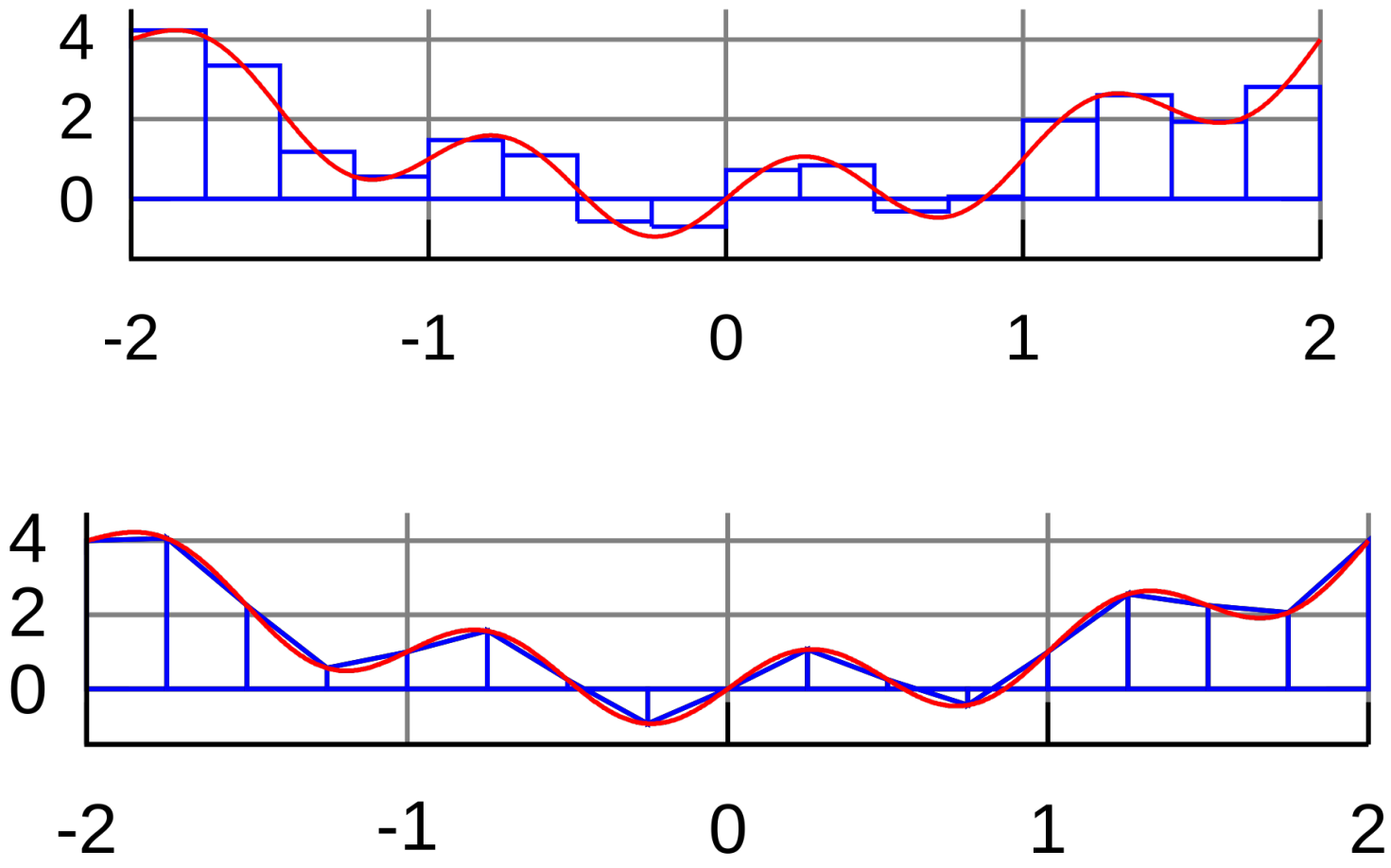
- Integration = calculation of area as a function
- Solving integral or differential equations is a key problem
- Its also very difficult and many different methods are needed

A first nonexample



- An ellipse is a pretty simple shape
- There is no simple way to express the perimeter of the ellipse in terms of elementary functions (algebraic functions, exponential functions *etc.*)
- So we cannot hope to solve integrals in general

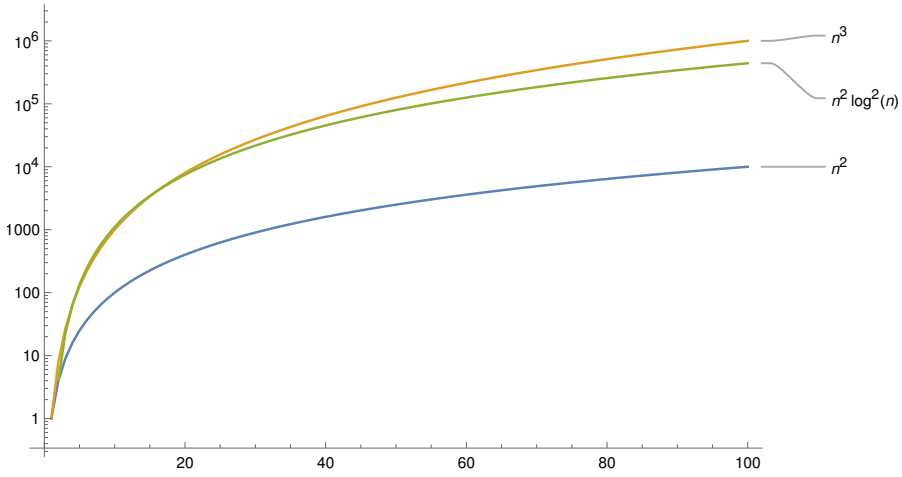
A first solution step: numerical integration

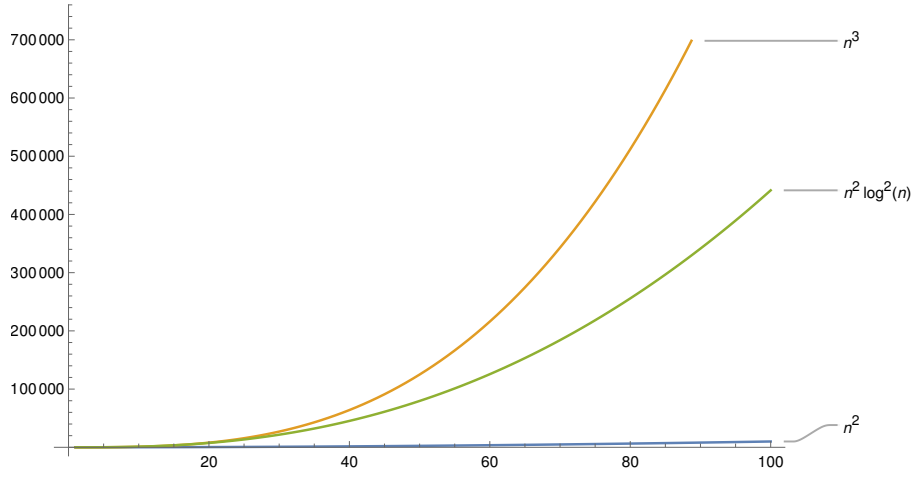


- Idea Ask for computer help
- Numerical solutions are quite powerful but not exact
- Question What about computer help to find exact solutions?

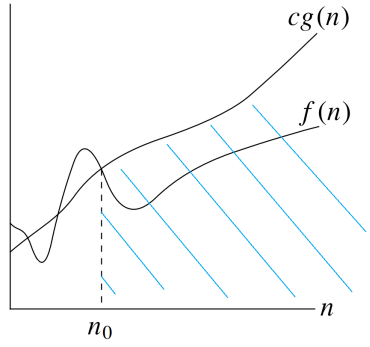
Enter, the theorem

Symbolic integration of rational functions can be done algorithmically in $O((n \log n)^2)$





- rational function = poly/poly, n=maximal involved degree, log is base two
- The above uses that $mult \in O(n \log n)$ (beefed-up Karatsuba)
- Landau–Bachmann notation:



Hopeless most of the time

```
Integrate[(5 + 6 x + 4 x^2 + 3 x^3 + 9 x^4) / (1 + x + x^2), x] //
 FullSimplify
```

$$x + 3 \cdot (-1 + x)\, x^2 - \sqrt{3}\, \text{ArcTan}\left[\frac{1 + 2x}{\sqrt{3}}\right] + \frac{11}{2} \text{Log}\left[1 + x + x^2\right]$$

```
Integrate[(5 + 6 x + 4 x^2 + 3 x^3 + 9 x^4) / Sqrt[(1 + x + x^(1/2))],
  x] // FullSimplify
```

$$\frac{1}{3\,440\,640}\left(2\sqrt{1 + \sqrt{x} + x}\left(13\,238\,191 + 2 \cdot \left(-5\,902\,549\sqrt{x} + 5\,418\,164\,x - 580\,104\,x^{3/2} - 975\,552\,x^2 + 1\,342\,080\,x^{5/2} + 729\,600\,x^3 - 1\,827\,840\,x^{7/2} + 1\,720\,320\,x^4\right)\right) + 10\,372\,005\,\text{ArcSinh}\left[\frac{1 + 2\sqrt{x}}{\sqrt{3}}\right]\right)$$

- Risch's algorithm and friends can compute a variety of integrals
- However, there are huge limitations
- Wannabe theorem Almost no integral has a nice solution

Thank you for your attention!

I hope that was of some help.