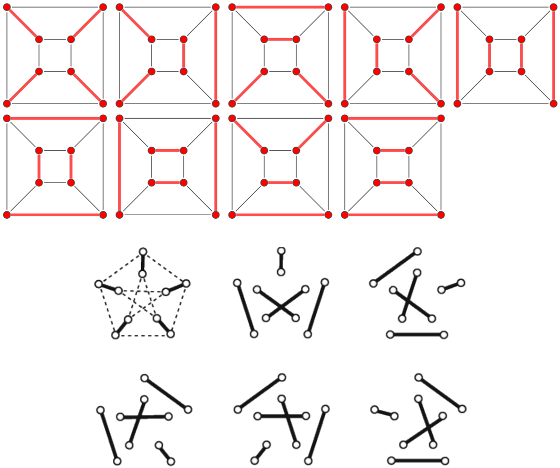
What is...Kasteleyn's theorem?

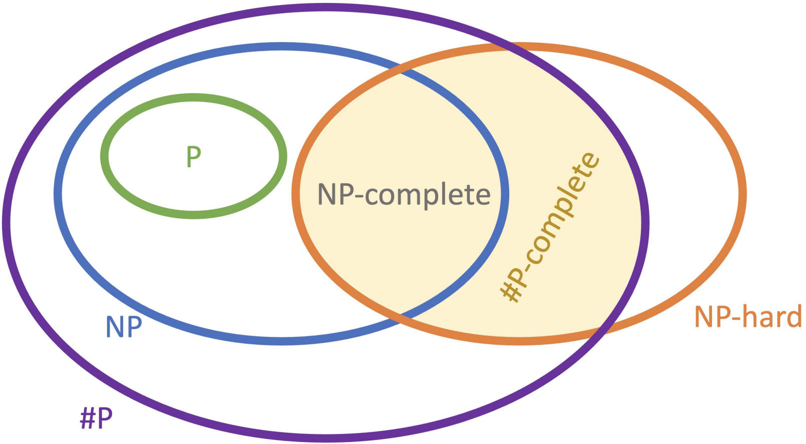
Or: Difficult, yet easy

Perfect matchings



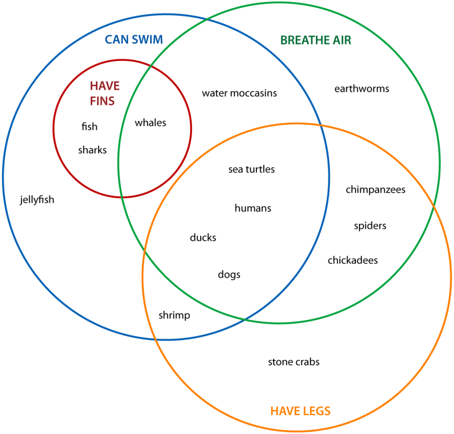
- Matching = pairing of all vertices
- Perfect matching = matching + edges are not adjacent
- Question Count perfect matchings!

A difficult problem



- Counting perfect matchings is #P complete
- For this video #P complete = very difficult

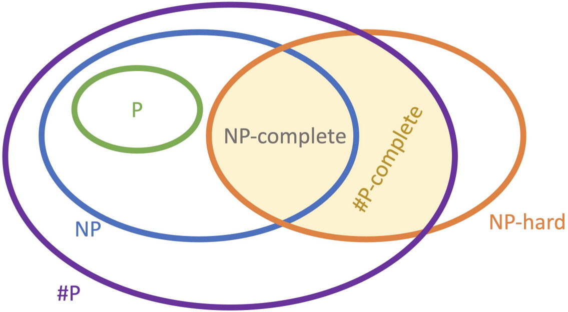
Or maybe not?



- ▶ Counting perfect matchings is #P complete in general
- ▶ That does not mean it is difficult for all graphs, e.g. for the ∞ subclass of edgeless graphs the count is easy (silly example)
- ▶ Task Find good subclasses for which this is easy

Enter, the theorem

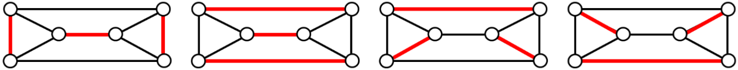
For planar graphs counting perfect matching is computable in polynomial time

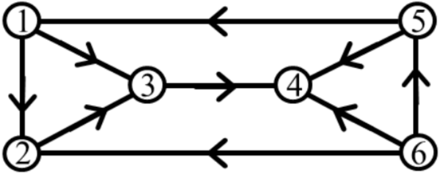


This remarkably fast

- There are not many other classes of graphs where the counting can be done in polynomial time
- For example, for bipartite graphs one is already in #P

Use the adjacency matrix





$$A(G) = \begin{pmatrix} 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \\ 1 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ -1 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 \end{pmatrix}, \quad det = 16$$

- Fact We can orient the edges so that every face has an odd number of clockwise edges (can be done fast and algorithmically)
- Take the weighted adjacency matrix $A(G)$
- $\#\text{perfect matchings} = \sqrt{det\, A(G)}$

Thank you for your attention!

I hope that was of some help.