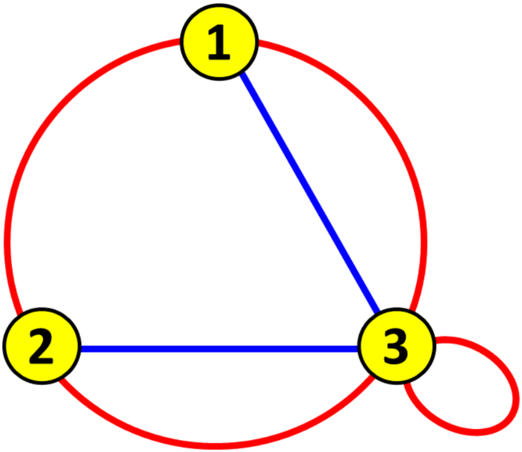
What is...the commuting matrix theorem?

Or: Around 25%

Two graphs, many paths

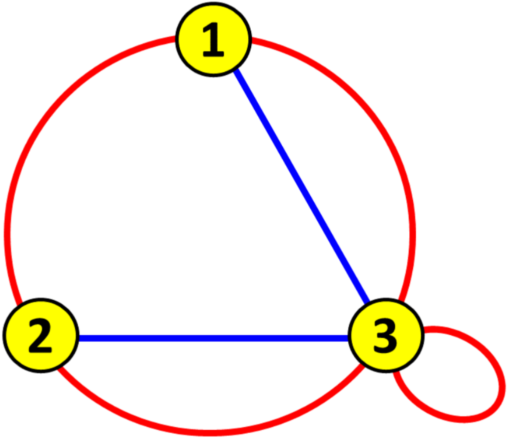


- Setting We have k graphs with n vertices; here X=red and Y=Blue
- Observe that # Red-then-blue paths = # Blue-then-red paths
- Question What is the maximal k such that all coloring path countings work?

Adjacency matrices

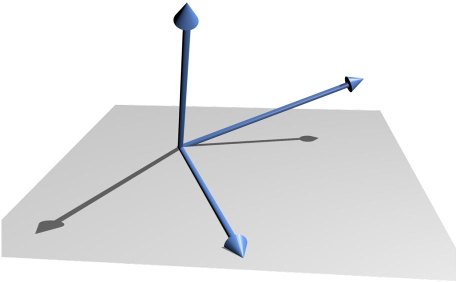
$$X = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$



- **Setting** Take k nxn matrices, here X and Y as above
- Observe that $XY = YX$, $XXY = XYX = YXX$ etc.
- **Question** What is the maximal k such that we have k commuting matrices?

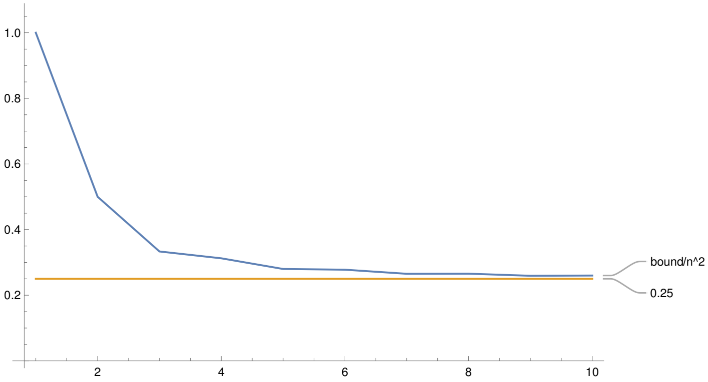
"How many" – better formulated



- We have infinitely many commuting matrices, namely M, $2M$, $3M$, ...
- Better: Only consider linearly independent matrices
- Question What is the maximal k such that we have k commuting linearly independent matrices?

Enter, the theorem

Over any field, there are at most $\lfloor n^2/4 \rfloor + 1$ commuting nxn matrices



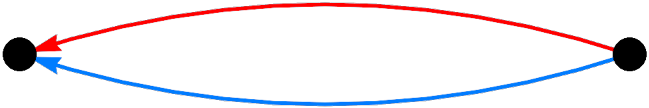
Thus, the maximal dimensions is about 25% of the whole space

- The bound is achieved (next slide)
- A bit counterintuitive: for large n, pick two nxn matrices "at random" and they commute with probability ≈ 0

25% commuting matrices

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$Y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$



- The matrices $\left(\begin{smallmatrix} 0 & X \\ 0 & 0 \end{smallmatrix}\right)$ all commute
- Choose one of the entries of X to have value 1 and the rest zero $\Rightarrow \lfloor n^2/4 \rfloor$
- The $+1$ corresponds to the identity matrix

Thank you for your attention!

I hope that was of some help.