What is...the choice zoo?

Or: No general choice, please!

The setting

> *It is a peculiar fact that all the transfinite axioms are deducible from a single one, the axiom of choice, — the most challenged axiom in the mathematical literature.*
>
> D. Hilbert (1926)

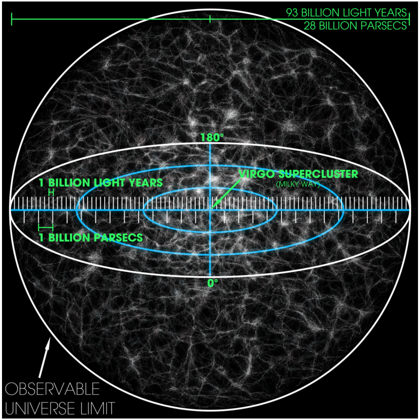
> *It is the great and ancient* ***problem of existence*** *that underlies the whole controversy about the axiom of choice.*
>
> W. Sierpiński (1958)

> *The Axiom of Choice has easily the most tortured history of all the set–theoretic axioms.*
>
> Penelope Maddy (Believing the axioms I)[1]

> *Of course not, but I am told it works even if you don't believe in it.*
>
> Niels Bohr (when asked whether he really believed a horseshoe hanging over his door would bring him luck).[2]

- ZF = Zermelo–Fraenkel set theory = standard set axioms for some people
- ZFC = ZF + axiom of choice (AC) = standard set axioms for some other people
- Question AC is a bit "weird", so it what sense does one need AC?

“all” is rather large



- **AC** The Cartesian product of nonempty sets is nonempty (an element of that product is a choice function)
- **‘Problem’** Most sets are “way too big” (at least for my brain) to be of any use
- **Idea** Weaken AC and see whether we can still describe “enough interesting math”

Some first weaker versions

AC: $\prod_{i \in I} X_i \neq \emptyset$ whenever all $X_i \neq \emptyset$

Definition 2.9. *1.* $\mathbf{CC}(\mathbb{R})$ *states that for each sequence* $(X_n)_{n\in\mathbb{N}}$ *of non–empty subsets* X_n *of* $\mathbb{R}$*, the product set* $\prod_{n\in\mathbb{N}} X_n$ *is non–empty.*

2. $\mathbf{CC}(\mathbb{Z})$ *states that for each sequence* $((X_n, \leq_n))_{n\in\mathbb{N}}$*, each* $(X_n, \leq_n)$ *being order–isomorphic to the ordered set of integers, the product set* $\prod_{n\in\mathbb{N}} X_n$ *is non–empty.*

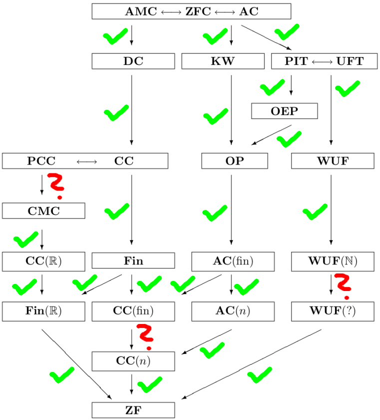
3. $\mathbf{CC}(\mathrm{fin})$ *states that for each sequence* $(X_n)_{n\in\mathbb{N}}$ *of non–empty, finite sets, the product set* $\prod_{n\in\mathbb{N}} X_n$ *is non–empty.*

4. $\mathbf{CC}(n)$*, for* $n \in \mathbb{N}^+$*, states that for each sequence* $(X_n)_{n\in\mathbb{N}}$ *of* n*–element sets, the product set* $\prod_{n\in\mathbb{N}} X_n$ *is non–empty.*

- We have two types of sets: the index set I and the X_i
- We can restrict the types of I or X_i
- **Example ($\mathbf{CC}(\mathbb{R})$).** Restrict I to $\mathbb{N}$ and X_i to $X_i \subset \mathbb{R}$
- Question In what sense are all the various versions of AC related?

Enter, the theorem

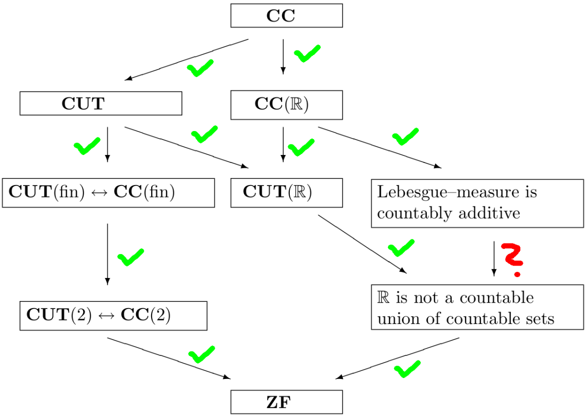
Here is a zoo between ZF and ZFC:



All of these give strictly different set theories with the three exceptions CC → CMC, CC(fin) → CC(n) and WUF(ℕ) → WUF(?) where we do not know (in 2023)

I will describe a subtree on the next slide

More refinements



- CUT = the Countable Union Theorem = countable unions of at most countable sets are at most countable
- CUT(X) = restriction to subsets of $\mathbb{R}$, finite sets or 2-elements sets
- All of these give strictly different set theories with the marked exception where we do not know (in 2023)

Thank you for your attention!

I hope that was of some help.