# What is...the exterior algebra?

Or: Anticommuting polynomials.

## Commute vs. anticommute

The polynomial algebra

$$\mathbb{R}[X_1, X_2, X_3]$$
$$= \mathbb{R}\langle X_1, X_2, X_3\rangle/(X_iX_j = X_jX_i)$$

Variables commute

The exterior algebra

$$\mathrm{Ext}(X_1, X_2, X_3)$$
$$= \mathbb{R}\langle X_1, X_2, X_3\rangle/(X_iX_j = -X_jX_i)$$

Variables anticommute

Let us multiply two polynomials:

| $\cdot$ | $b_1X_1$ + | $b_2X_2$ + | $b_3X_3$ |
|---|---|---|---|
| $a_1X_1$ + | $a_1b_1X_1X_1$ | $a_1b_2X_1X_2$ | $a_1b_3X_1X_3$ |
| $a_2X_2$ + | $a_2b_1X_2X_1$ | $a_2b_2X_2X_2$ | $a_2b_3X_2X_3$ |
| $a_3X_3$ | $a_3b_1X_3X_1$ | $a_3b_2X_3X_2$ | $a_3b_3X_3X_3$ |

$$a_1b_1X_1^2 + a_2b_2X_2^2 + a_3b_3X_3^2$$
$$+(a_1b_2 + a_2b_1)X_1X_2$$
$$+(a_1b_3 + a_3b_1)X_1X_3$$
$$+(a_2b_3 + a_3b_2)X_2X_3$$

$$0$$
$$+(a_1b_2 - a_2b_1)X_1X_2$$
$$+(a_1b_3 - a_3b_1)X_1X_3$$
$$+(a_2b_3 - a_3b_2)X_2X_3$$

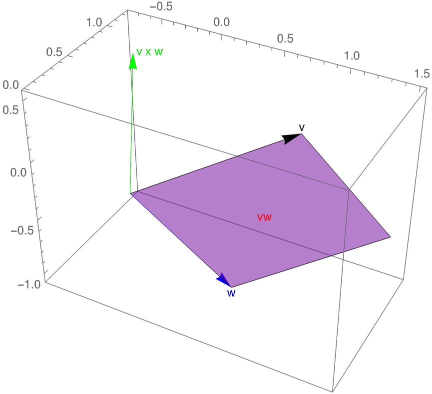
# Higher dimensional vectors

These are the same coefficients as for the cross product:

$$\begin{aligned}&(a_1 b_2 - a_2 b_1) X_1 X_2\\+&(a_1 b_3 - a_3 b_1) X_1 X_3\\+&(a_2 b_3 - a_3 b_2) X_2 X_3\end{aligned}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} x \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 b_2 - a_2 b_1 \\ a_1 b_3 - a_3 b_1 \\ a_2 b_3 - a_3 b_2 \end{pmatrix}$$

However, the first is more like a 2-dimensional object:



## Lets count dimensions

Write $\mathrm{Ext}^k(X_1, X_2, X_3)$ for polynomials of degree $k$ in $\mathrm{Ext}(X_1, X_2, X_3)$.

- $\mathrm{Ext}^0(X_1, X_2, X_3)$ is spanned by $\{1\}$
  $\dim \mathrm{Ext}^0(X_1, X_2, X_3) = \binom{3}{0} = 1$
- $\mathrm{Ext}^1(X_1, X_2, X_3)$ is spanned by $\{X_1, X_2, X_3\}$
  $\dim \mathrm{Ext}^1(X_1, X_2, X_3) = \binom{3}{1} = 3$
- $\mathrm{Ext}^2(X_1, X_2, X_3)$ is spanned by $\{X_1X_2, X_1X_3, X_2X_3\}$
  $\dim \mathrm{Ext}^2(X_1, X_2, X_3) = \binom{3}{2} = 3$
- $\mathrm{Ext}^3(X_1, X_2, X_3)$ is spanned by $\{X_1X_2X_3\}$
  $\dim \mathrm{Ext}^3(X_1, X_2, X_3) = \binom{3}{3} = 1$
- All others are zero and the total dimension is $2^3 = 8$
  $\dim \mathrm{Ext}(X_1, X_2, X_3) = 2^n$

In general $\dim \mathrm{Ext}^k(X_1, ..., X_n) = \binom{n}{k}$ and $\dim \mathrm{Ext}(X_1, ..., X_n) = 2^n$

## For completeness: A formal definition.

The exterior algebra $\mathrm{Ext}(V)$ of a vector space $V$ over a field (say not of characteristic 2) is defined as the quotient algebra of the tensor algebra $\mathrm{T}(V)$ by the two-sided ideal $I$ generated by the relation

$$X \otimes Y = -Y \otimes X$$

- Very often one writes *e.g.* $X \wedge Y$ for the image of $X \otimes Y$ under the canonical surjection $\mathrm{T}(V) \twoheadrightarrow \mathrm{Ext}(V)$
- Note that $X \wedge Y = -Y \wedge X$ implies $X \wedge X = -X \wedge X$. Thus, $X \wedge X = 0$ in $\mathrm{Ext}(V)$
- If we choose a basis $\{X_i\}$ of $V$, then $\mathrm{Ext}(V)$ is the polynomial ring in non-commuting variables $\{X_i\}$

## Here is the determinant

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

| $\cdot$ | $bX_1$ + | $dX_2$ |
|---|---|---|
| $aX_1$ + | $abX_1X_1$ | $adX_1X_2$ |
| $cX_2$ | $bcX_2X_1$ | $cdX_2X_2$ |

$$0$$

$$(ad - bc)X_1X_2$$

Using $\wedge$:

$$\begin{pmatrix} a \\ c \end{pmatrix} \wedge \begin{pmatrix} b \\ d \end{pmatrix} = \left(a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \wedge \left(b\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$= ab\begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix} + ad\begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \end{pmatrix} + bc\begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \end{pmatrix} + cd\begin{pmatrix} 0 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= (ad - bc)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (ad - bc)X_1 \wedge X_2$$

det is the scalar in front of $X_1 \wedge \ldots \wedge X_n \in \mathrm{Ext}(X_1, \ldots, X_n)$

**Thank you for your attention!**

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I hope that was of some help.