What is...the dual vector spaces?

Or: Two flipped sides.

My wish list for duality (in mathematics and beyond).

V is some object, V^* its dual. The duality $\_^*$ should satisfy:

- I want $(V^*)^* \cong V$
- I want that an operation $V \to W$ turns into an operation $W^* \to V^*$
- I want that operations $V \to W^*$ correspond to operations $W^* \to V$

Duality in mathematics is not a theorem, but a "principle" (Atiyah)

Everything that is true about V has a dual statement which is true about V^*

Two points meet in one line, two lines meet in one point:

Strictly speaking this needs projective geometry...

Let us look at linear maps

Consider the usual inner product on $\mathbb{R}^3$:

$$\langle v, w\rangle = v_1 w_1 + v_2 w_2 + v_3 w_3$$

Another way of writing this:

$$\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$\langle v, \_\rangle \colon \mathbb{R}^3 \to \mathbb{R}, \quad w \mapsto \langle v, w\rangle \leftrightsquigarrow \begin{pmatrix} \langle v,(1,0,0)\rangle & \langle v,(0,1,0)\rangle & \langle v,(0,0,1)\rangle \end{pmatrix} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

Each vector $v \in \mathbb{R}^3$ corresponds to a linear map $\langle v, \_\rangle \colon \mathbb{R}^3 \to \mathbb{R}$ and *vice versa*

The transpose space

$\mathbb{R}^2$ as column vectors

$$\mathbb{R}^2 = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : \mathbb{R} \to \mathbb{R}^2 \mid a, b \in \mathbb{R} \right\}$$

$\mathbb{R}^2$ as row vectors

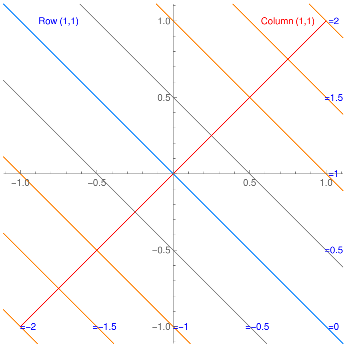
$$(\mathbb{R}^2)^* = \left\{ \begin{pmatrix} a & b \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R} \mid a, b \in \mathbb{R} \right\}$$

The difference?

$\begin{pmatrix} a \\ b \end{pmatrix}$ takes a number λ and gives a vector $\lambda \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{pmatrix} a & b \end{pmatrix}$ takes a vector $\begin{pmatrix} c \\ d \end{pmatrix}$ and gives a number $ac + bd$

Example ($a = b = 1$). One of them is one-dimensional, the other kind of as well



For completeness: A formal definition.

The dual vector space is $V^* = \mathrm{End}_K(V, \mathbb{K})$

Important facts for V finite-dimensional be careful with infinities:

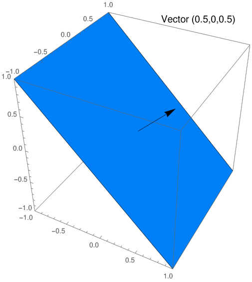
- Let $B = \{v_1, ..., v_n\}$ be a basis of vector space V. The dual vector space $V^* = \mathrm{End}_K(V, \mathbb{K})$ has basis $B = \{v_1^*, ..., v_n^*\}$ given by linear maps

$$v_i^*\colon V \to \mathbb{K}, \quad v_i^*(v_j) = \begin{cases} 1 & \text{if i=j} \\ 0 & \text{else} \end{cases}$$

- $(V^*)^* \cong V$ canonically by $v^{**}(\phi) = \phi(v)$
- $f\colon V \to W$ gives $f^*\colon W^* \to V^*$ by $f(\phi) = \phi \circ f$
- $f\colon V \to W^*$ is the same as $f^*\colon W \to V^*$ by $f(v) = w^* \Leftrightarrow f^*(w) = v^*$

Dimensions turn around

$$v = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} \leftrightsquigarrow$$



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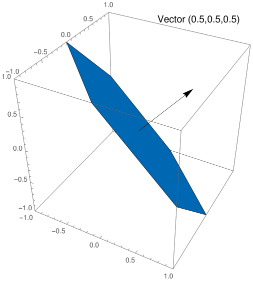
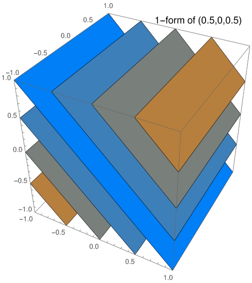
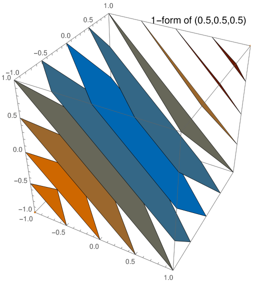


$$\langle v, \_ \rangle \leftrightsquigarrow$$



$$\langle v, \_ \rangle \leftrightsquigarrow$$



Thank you for your attention!

I hope that was of some help.