What is...Sylvester's law of inertia?

Or: Signatures is linear algebra

The Gram matrix

$$A = \begin{pmatrix} 1 & -1 \\ -1 & -2 \end{pmatrix} \leftrightsquigarrow \langle v, w \rangle = 1 \cdot v_1 w_1 + -1 \cdot v_1 w_2 + -1 \cdot v_2 w_1 + -2 \cdot v_2 w_2$$

$$B = \begin{pmatrix} -8 & -2 \\ -2 & 1 \end{pmatrix} \leftrightsquigarrow \langle v, w \rangle = -8 \cdot v_1 w_1 + -2 \cdot v_1 w_2 + -2 \cdot v_2 w_1 + 1 \cdot v_2 w_2$$

$$B \neq PAP^{-1} \text{ since } \mathrm{tr}(A) = -1 \neq -7 = \mathrm{tr}(B)$$

Question. How can one decide whether A and B describe the same inner product?

Matrices relations $A \sim B$

A equivalent to $B \Leftrightarrow B = PAQ^{-1}$

$\Leftrightarrow$ same linear maps $V \to W$ up to the choice of a pair of bases

$\Leftrightarrow$ they have the same rank

A similar to $B \Leftrightarrow B = PAP^{-1}$

$\Leftrightarrow$ same linear maps $V \to V$ up to the choice of a basis

$\Leftrightarrow$ they have the same Jordan normal form (when we work over $\mathbb{C}$)

A congruent to $B \Leftrightarrow B = PAP^T$

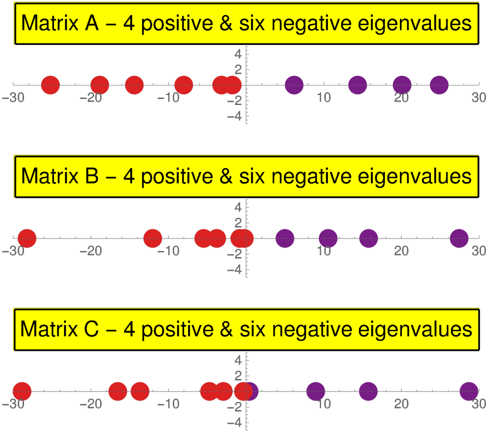
$\Leftrightarrow$ same bilinear form up to the choice of a basis

$\Leftrightarrow$ here comes Sylvester (for certain A, B)

Count positive and negative eigenvalues

"Random" symmetric matrix A, "random" matrices P, Q, $B = PAP^T$, $C = QAQ^T$.

First thing to check: eigenvalues!



The invariant we are looking for is the signature:

$$\#\{\text{positive eigenvalues}\} - \#\{\text{negative eigenvalues}\}$$

For completeness: A formal statement

Two symmetric matrices $A, B \in \mathrm{Mat}_{n\times n}(\mathbb{R})$ have the same number of positive n_+, negative n_- and zero n_0 eigenvalues if and only if they are congruent

n_+, n_- and n_0 are invariants under congruence

Important facts:

(a) There is an analog statement over $\mathbb{C}$

(b) For invertible matrices $\mathrm{sgn}(A) = n_+ - n_-$ is a complete invariant under congruence The signature

(c) A normal form under congruence is

$$\left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Why on earth is this called "Sylvester's law of inertia"?

Sylvester's love of poetry and language manifested itself in notable ways even in his mathematical writings. His mastery of French, German, Italian, and Greek was often reflected in the mathematical neologisms - like "meicatecticizant" and "tamisage" – for which he gained a certain notoriety. Moreover, literary illusions, poetic quotations, and unfettered hyperbole spiced his published papers and lectures.

K.H. Parshall, James Joseph Sylvester, American National Biography 21 (Oxford, 1999), 226-228.

Inertia.—The unchangeable number of integers in the excess of positive over negative signs which adheres to a quadratic form expressed as the sum of positive and negative squares, notwithstanding any real linear transformations impressed upon such form.

Thank you for your attention!

I hope that was of some help.