What is...the Jordan normal form?

Or: Why (almost) all matrices are diagonalizable.

What are the equivalence classes under conjugation?

Consider a complex matrix:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Question. What are the complex matrices N such that $M = P^{-1}NP$ for some invertible complex matrix P?

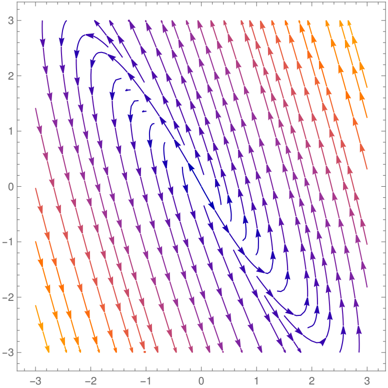
Answer. The matrices with the same Jordan-type.

A first example.

$M = \begin{pmatrix} -1+a & -1+a \\ 4-a & 3-a \end{pmatrix}$; eigenvalues are $1 \pm \sqrt{a}$.

For $a \neq 0$ we have two eigenvalues, so M is diagonalizable.

For $a = 0$ we get the sole (up to scalars) eigenvector $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$, so M is not diagonalizable.



What can we do?

What is the best approximation to a diagonal matrix? Jordan blocks!

$$J_\lambda = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Eigenvalues are all λ, the only eigenvector is $(1, 0)$.

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$$J_\lambda = \begin{pmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

Eigenvalues are all λ, the only eigenvector is $(1, 0, 0, 0)$.

We can't do better, so let us except this is the easiest possible.

For completeness: A formal definition.

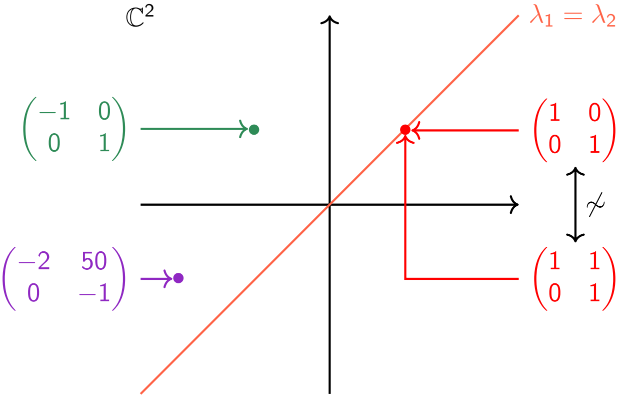
The Jordan normal form of M is a matrix equivalent to it of the form

$$\begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_k \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

The Jordan normal form exists, is unique up to order of the Jordan blocks J_i, and a complete invariant under base-change.

Almost all matrices are diagonalizable.

Knowing the eigenvalues and the size of the Jordan blocks determines matrices up to base-change $\sim$.



A random 2x2 matrix is almost always diagonalizable.

Thank you for your attention!

I hope that was of some help.