What are...traces?

Or: The best? matrix invariant

A silly looking, but powerful invariant

$$\mathrm{tr}\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}\right) = 1 + 5 + 9 = 15$$

$$\mathrm{tr}(A) = \sum \text{eigenvalues}, \quad \det(A) = \prod \text{eigenvalues}$$

(a) $\mathrm{tr}(\lambda \cdot A) = \lambda \cdot \mathrm{tr}(A)$, $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ Linear

(b) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ Cyclic

Traces generalize dimensions

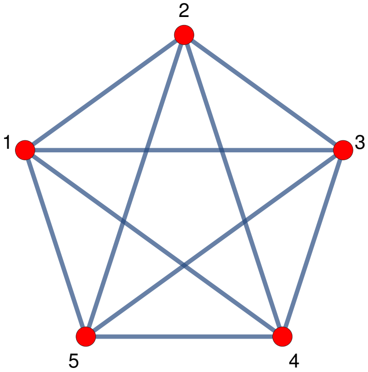
$$\mathrm{id}_3\colon \mathbb{R}^3 \to \mathbb{R}^3 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathrm{tr}(\mathrm{id}_3) = 3 = \dim \mathbb{R}^3$$

$$\mathrm{pr}_3^2\colon \mathbb{R}^3 \to \mathbb{R}^2 \leftrightsquigarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{tr}(\mathrm{pr}_3^2) = 2 = \dim \mathbb{R}^2$$

$$A = \begin{pmatrix} 3 & -2 & -4 \\ 1 & 0 & -2 \\ 1 & -1 & -1 \end{pmatrix}, \quad A^2 = A, \quad \mathrm{tr}(A) = 2 = \dim \mathbb{R}^2$$

For every projection $A^2 = A$ the trace is the dimension of the target

Applications? Sure!



$$\rightsquigarrow A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Question. How many triangles does this graph have?

$$A^3 = \begin{pmatrix} 12 & 13 & 13 & 13 & 13 \\ 13 & 12 & 13 & 13 & 13 \\ 13 & 13 & 12 & 13 & 13 \\ 13 & 13 & 13 & 12 & 13 \\ 13 & 13 & 13 & 13 & 12 \end{pmatrix}, \quad \mathrm{tr}(A^3) = 60$$

The graph has $\mathrm{tr}(A^3)/6$ triangles!

This works in general

For completeness: A formal statement

The trace $\mathrm{tr}\colon \mathrm{Mat}_{n \times n}(\mathbb{R}) \to \mathbb{R}$ is the (up to scalars) unique map satisfying

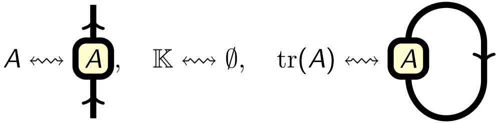
(a) $\mathrm{tr}(\lambda \cdot A) = \lambda \cdot \mathrm{tr}(A)$, $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$ Linear

(b) $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ Cyclic

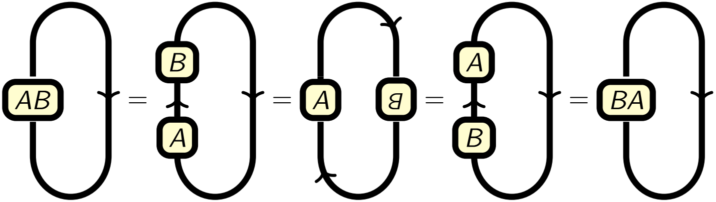
Important facts:

- The trace is basis independent $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(AP^{-1}P) = \mathrm{tr}(A)$, so can be defined for linear operators as well
- The trace of the identity matrix is the dimension
- The trace is the sum of the eigenvalues
- The trace is the sum of the diagonal elements

A picture is worth a thousand words



Cylcic:



This diagrammatic approach generalizes beyond the realm of matrices

Thank you for your attention!

I hope that was of some help.