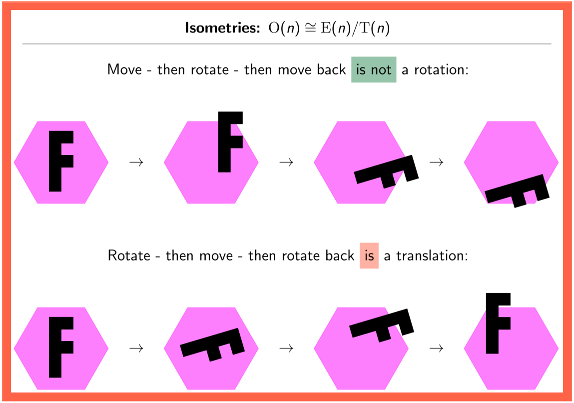
Lie theory - part 9

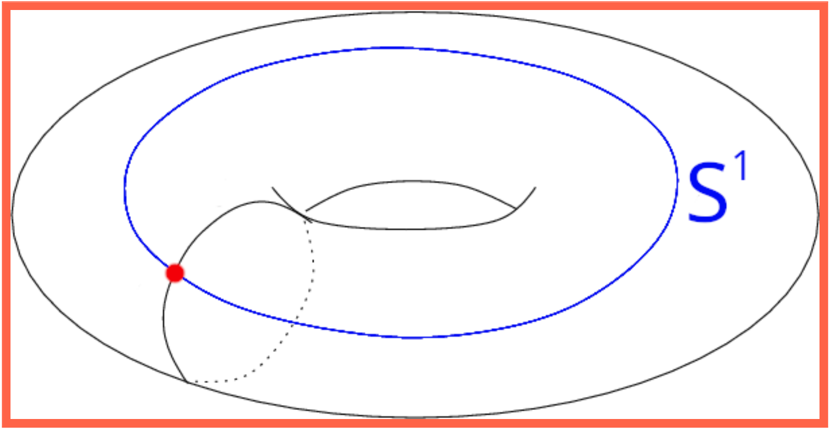
Or: Ideals and quotient Lie algebras

Ideals: the subalgebras you can safely "mod out"



- **Definition** An ideal $\mathfrak{a} \triangleleft \mathfrak{g}$ satisfies $[X, A] \in \mathfrak{a}$ for all $X \in \mathfrak{g}$, $A \in \mathfrak{a}$
- **Symmetry meaning** $\mathfrak{a}$ is a "direction" that the whole algebra can push you into, but you never leave
- **Analogy** Ideals are the Lie algebra version of normal subgroups (the ones compatible with quotients)

First (non-)examples

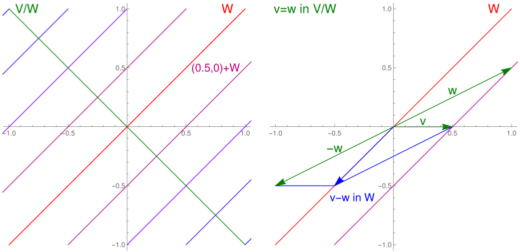


- Always $\{0\}$ and $\mathfrak{g}$ are ideals; $\ker(\varphi)$ is an ideal for any homomorphism φ
- Matrix example In $\mathfrak{gl}_n$, scalar matrices form a central ideal (they commute with everything)
- Non-example In $\mathfrak{gl}_2$, the subalgebra $\mathfrak{k} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} : a \in \mathbb{C} \right\}$ is not an ideal

Quotients: collapsing "invisible" directions

Linear identification along codim 1

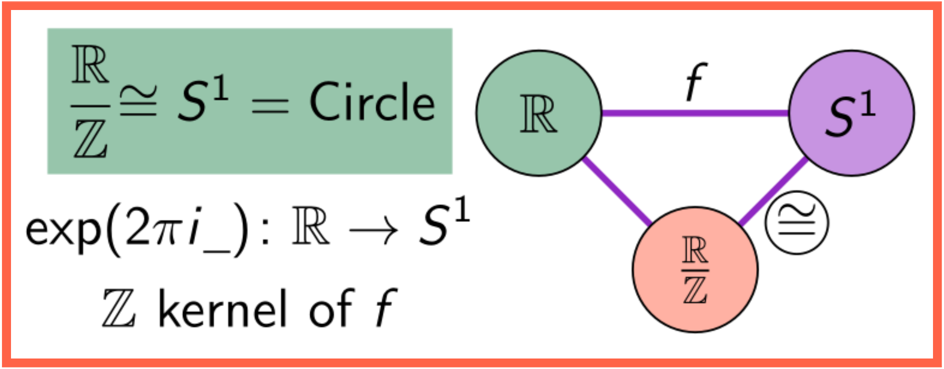
What happens if we collapse a line $W = \mathbb{R}(1,1)$ in $V = \mathbb{R}^2$ to a point?



The lines parallel to W are the points of V/W, $\dim V/W = 1$

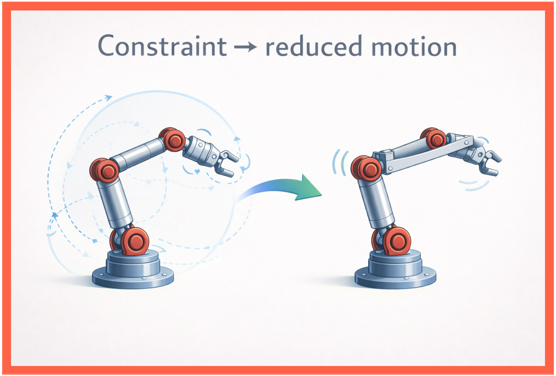
- Vector spaces The quotient $\mathfrak{g}/\mathfrak{a}$ identifies elements differing by things in $\mathfrak{a}$
- Bracket on cosets Define $[X + \mathfrak{a},\ Y + \mathfrak{a}] = [X, Y] + \mathfrak{a}$
- Why ideals This is well-defined exactly because $\mathfrak{a}$ is stable under bracketing with all of $\mathfrak{g}$

The first isomorphism theorem (Lie version)



- Statement If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism, then $\mathfrak{g}/\ker(\varphi) \cong \mathrm{im}(\varphi)$
- Meaning "Forget what φ kills" and you get exactly the symmetry you can actually see in the target
- Use This is the basic tool for simplifying Lie algebras by factoring out redundant directions

Back to symmetries: constraints and reduction



- Constraints Often you impose relations that declare some motions “invisible” $\Rightarrow$ quotient by an ideal
- Reduction You keep the symmetry information that still acts nontrivially after the constraint
- Preview Next: simple/semisimple Lie algebras = those with no nontrivial solvable ideals (structure theory begins)

Thank you for your attention!

Next time: The centre, derived algebra, and “simple” vs “solvable”