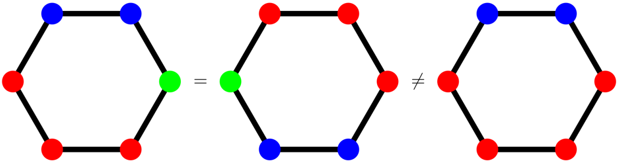
Lie theory - part 8

Or: Homomorphisms and subalgebras

Lie algebra homomorphisms

- Definition A linear map $\varphi : \mathfrak{g} \to \mathfrak{h}$ with $\varphi([X, Y]) = [\varphi(X), \varphi(Y)]$
- Matrix intuition In $\mathfrak{gl}_n$: $[X, Y] = XY - YX$, so "respecting the bracket" means respecting commutators
- These preserve infinitesimal symmetry (not just vector space structure)

Subalgebras



- The rotation group $G = \{1, 60^\circ, \dots\}$ of order 6 acts on necklaces
- $|\mathrm{fix}(1)| = 3^6$, $|\mathrm{fix}(60^\circ)| = 3$, $|\mathrm{fix}(120^\circ)| = 3^2$, $|\mathrm{fix}(180^\circ)| = 3^3$, $|\mathrm{fix}(240^\circ)| = 3^2$, $|\mathrm{fix}(300^\circ)| = 3$
- Thus, there are

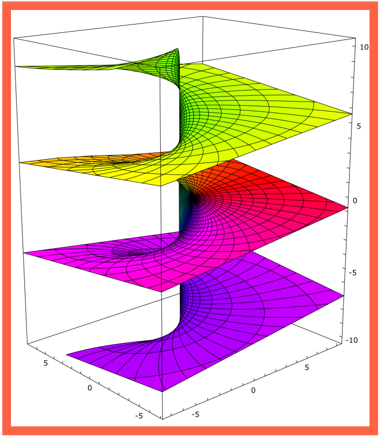
$$\frac{1}{6}\sum_{g\in G}|\mathrm{fix}(g)| = \frac{1}{6}(3^6 + 3^3 + 2\cdot 3^2 + 2\cdot 3) = 130$$

necklaces

- Definition A subspace $\mathfrak{k} \subseteq \mathfrak{g}$ with $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$
- Examples Diagonal matrices; upper triangular matrices; $\mathfrak{so}_n \subseteq \mathfrak{gl}_n$
- Mental model "Stay inside after taking commutators" = "closed under infinitesimal motion + swapping"

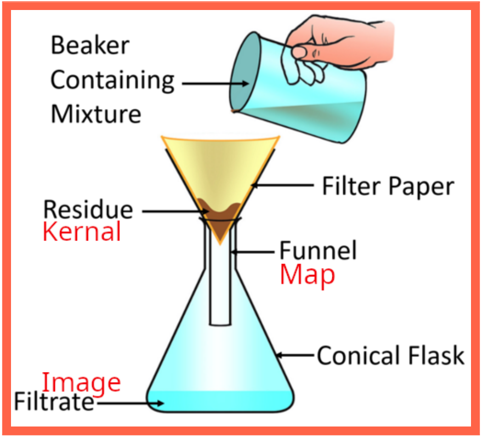
From subgroups to subalgebras (and the subtle reverse direction)

Branches of log:



- Easy If $H \leq G$ is a Lie subgroup, then $T_eH \subseteq T_eG$ is a Lie subalgebra
- Commutators in H stay in H, so their infinitesimal shadows stay in T_eH
- Subtlety Given $\mathfrak{k} \subseteq \mathfrak{g}$, there is a connected subgroup "generated by $\mathfrak{k}$", but it may be immersed rather than nicely embedded

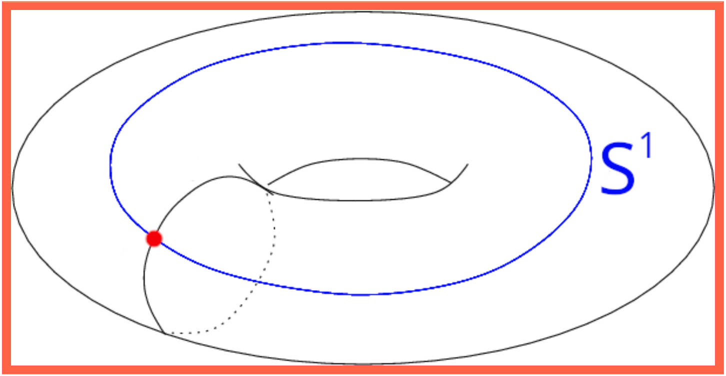
Kernel and image: ideals appear automatically



- **Image** If $\varphi : \mathfrak{g} \to \mathfrak{h}$ is a homomorphism, then $\mathrm{im}(\varphi)$ is a Lie subalgebra of $\mathfrak{h}$
- **Kernel** $\ker(\varphi)$ is an *ideal*: $[X, K] \in \ker(\varphi)$ for all $X \in \mathfrak{g}$, $K \in \ker(\varphi)$
- **Preview** Ideals are exactly the "normal subalgebras" that let you form quotients $\mathfrak{g}/\mathfrak{a}$

Why care: homomorphisms/subalgebras = symmetry in disguise

$S^1 \subset S^1 \times S^1$:



- The meta-idea A Lie algebra is "infinitesimal symmetry"; a homomorphism is a symmetry-preserving change of coordinates
- Subalgebras = allowed motions Choosing $\mathfrak{k} \subseteq \mathfrak{g}$ means: we restrict to a smaller symmetry world (constraints, invariants, conserved directions)
- Back to the slogan "Differentiate symmetry": $G \to \mathfrak{g}$; "Compare symmetries": maps/subalgebras

Thank you for your attention!

Next time: “Integrate back” via subgroups and quotients