

Lie theory - part 28

Or: The Weyl group

A change of basis that swaps the diagonal

$$\begin{array}{ccc} \pi: \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix} & \longleftrightarrow & R_\pi: \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ \updownarrow & & \updownarrow \\ C_\pi: \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \longleftrightarrow & \pi^{-1}: \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \end{array}$$

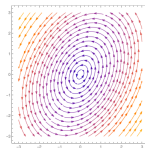
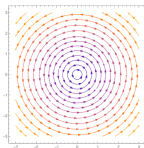
- **Start** In $G = SU(2)$, take $D = \text{diag}(i, -i)$
- **Change of basis** $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ has determinant 1 and belongs to $SU(2)$
- **Conjugate** $sDs^{-1} = \text{diag}(-i, i)$: the diagonal entries exchange places

Which matrices keep the diagonal diagonal?

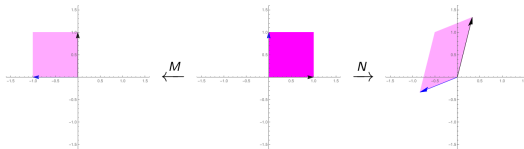
Twice the same beast?

A rotation! $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Another rotation? $N = \begin{pmatrix} 1/3 & -5/6 \\ 4/3 & -1/3 \end{pmatrix}$



The action is similar:



- **Torus** $T = \{\text{diag}(z, z^{-1}) : |z| = 1\}$
- **Normalizer** $g \in N_G(T)$ means gAg^{-1} stays diagonal for every $A \in T$
- **$SU(2)$** Diagonal or anti-diagonal matrices: $N_G(T) = T \sqcup sT$

Different matrices, the same swap

A game with strings

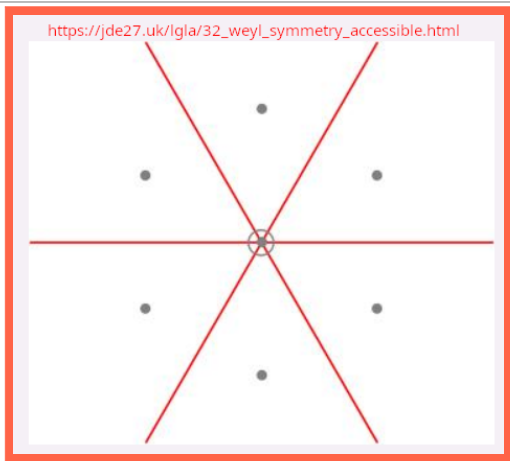
Connect eight points at the bottom with eight points at the top:

or

You just invented the symmetric group S_8

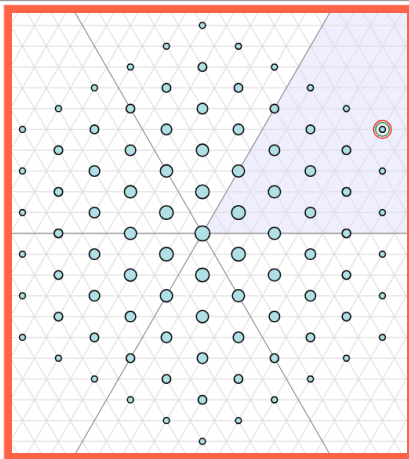
- ▶ **Example** s and $sD = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$ both swap the diagonal entries
- ▶ **One coset** Diagonal factors commute, so sT gives one swap
- ▶ **Quotient** $s^2 = -I \in T$, so $W = N_G(T)/T = \{T, sT\}$

$SU(3)$: three entries, six orders



- **Start** $D_3 = \text{diag}(1, i, -i)$ has three different diagonal entries
- **Swap** $r = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ gives $rD_3r^{-1} = \text{diag}(i, 1, -i)$
- **Weyl group** Six permutations of the entries, one coset each: $W \cong S_3$

The classification of Weyl groups



- **Permutations** $W(SU(n)) \cong S_n$: permute the n diagonal entries
- **Signs** B, C : all signed permutations, D : only even numbers of sign flips
- **All cases** Products of these and $W(E_6), W(E_7), W(E_8), W(F_4), W(G_2)$

Thank you for your attention!

Next time: Differential equations and symmetry