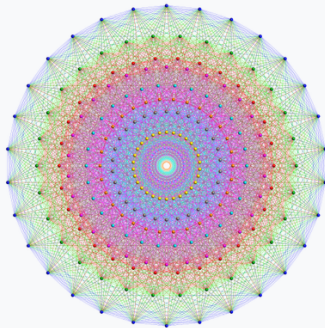


## Lie theory - part 27

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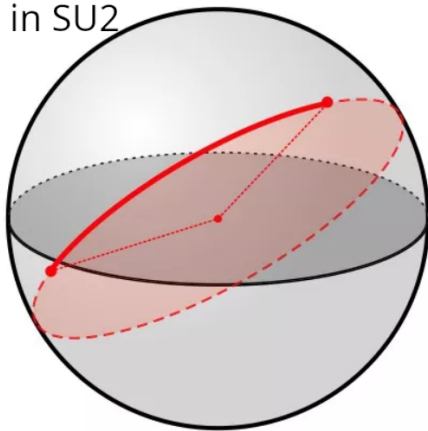
Or: Root systems

## Lie groups and Lie algebras



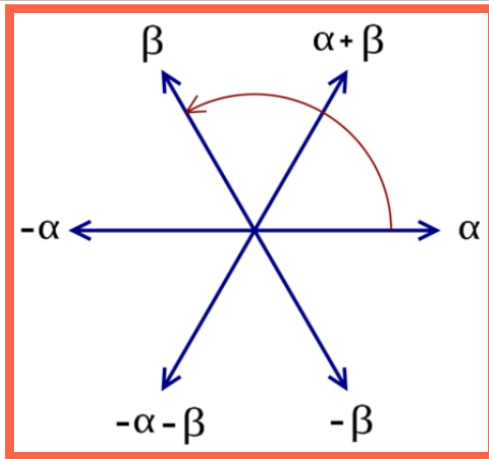
- 
- **Setting** Compact connected  $G$ , with a maximal torus  $T$
  - **Adjoint action** Let  $T$  act by conjugation on the complexified Lie algebra  $\mathfrak{g}_{\mathbb{C}}$
  - **Roots** The nonzero weights of this action form the root system

U1 in  $SU(2)$



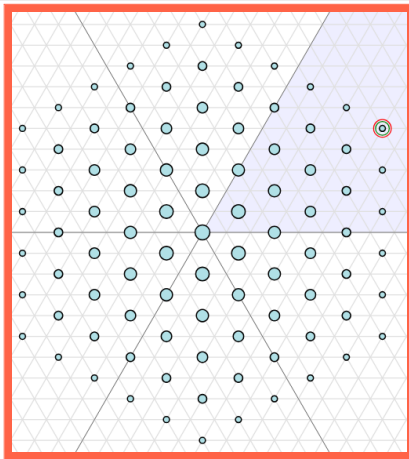
- Recall Our diagonal circle has weights  $+1, -1$  on  $\mathbb{C}^2$
- Conjugation On  $\mathfrak{g}_{\mathbb{C}}$ , the weights are  $0, +2, -2$
- Roots The two nonzero weights give opposite arrows on a line: type  $A_1$

## $SU(3)$ : a hexagon



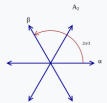
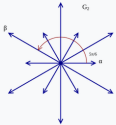
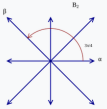
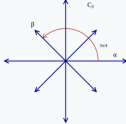
- **Diagonal torus** Write  $t = \text{diag}(z_1, z_2, z_3)$ , with  $|z_i| = 1$  and  $z_1 z_2 z_3 = 1$
- **One entry** Conjugation scales the matrix unit  $E_{ij}$  by  $z_i/z_j$ , for  $i \neq j$
- **Root picture** Six roots form a hexagon in two dimensions: type  $A_2$

## Roots move weights



- ▶ **Root vector** A vector in  $\mathfrak{g}_{\mathbb{C}}$  of weight  $\alpha$  sends  $V_{\lambda}$  into  $V_{\lambda+\alpha}$
- ▶  **$SU(2)$**  Raising and lowering shift weights by  $\pm 2$ , or give zero
- ▶ **Meaning** Roots describe the possible steps inside a representation

# The structure behind the picture

|                                                                                                                                  |                                                                                                                       |
|----------------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------|
| <p>Root system <math>A_1 \times A_1</math></p>  | <p>Root system <math>D_2</math></p>  |
| <p><math>A_2</math></p>                         | <p><math>G_2</math></p>              |
| <p>Root system <math>A_2</math></p>             | <p>Root system <math>G_2</math></p>  |
| <p><math>B_2</math></p>                         | <p><math>C_2</math></p>              |
| <p>Root system <math>B_2</math></p>             | <p>Root system <math>C_2</math></p>  |

- **Building blocks** Each opposite root pair gives a copy of  $\mathfrak{su}(2)$  inside  $\mathfrak{g}$
- **Rigid geometry** These copies force restricted angles and relative root lengths
- **Next** Reflections of the root picture lead to Weyl groups and classification

**Thank you for your attention!**

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Next time: Weyl groups and the idea of classification