

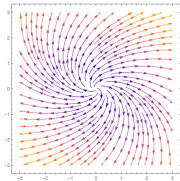
Lie theory - part 25

Or: Maximal tori and the diagonalization philosophy

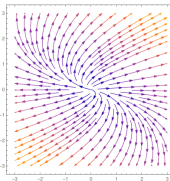
Commuting is the easy case

Are all matrices diagonalizable? Well, almost all...

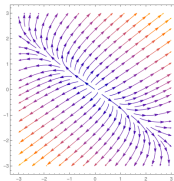
$$\begin{pmatrix} 1 & 1 \\ a & 1 \end{pmatrix} \begin{cases} \text{not dia over } \mathbb{R}, \text{ but over } \mathbb{C} \text{ if } a < 0, \\ \text{not dia over } \mathbb{R} \text{ or } \mathbb{C} \text{ if } a = 0, \\ \text{dia over } \mathbb{R} \text{ and } \mathbb{C} \text{ if } a > 0. \end{cases}$$



$a = -0.5$



$a = 0$

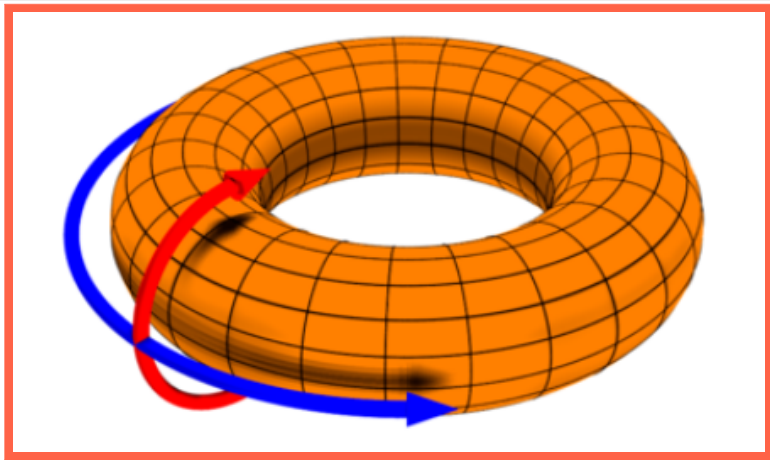


$a = 0.5$

Almost all matrices are diagonalizable – over $\mathbb{C}$ – we will see this when we generalize this notion to the Jordan normal form.

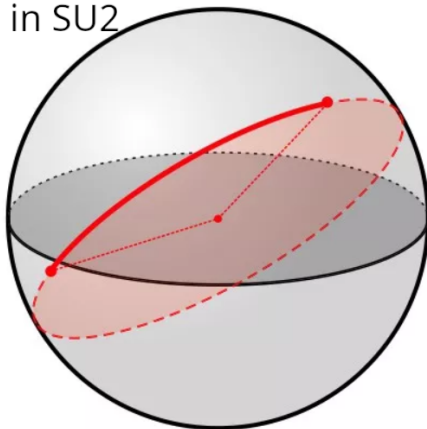
- **Easy case** Commuting unitary matrices can be diagonalized together
- **On the diagonal** Complicated matrices become phases on coordinate lines
- **Philosophy** Find the largest commuting piece and study the group there

So what is a torus?



- ▶ **Torus** A compact connected abelian Lie group: $T \cong (S^1)^{\times r}$
- ▶ **Complex** Diagonal matrices with entries $e^{i\theta_1}, \dots, e^{i\theta_n}$ form a torus
- ▶ **Real** Independent rotations in perpendicular planes form a torus in $SO(n)$

U1 in SU2

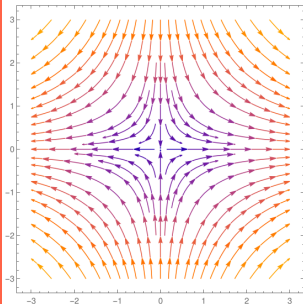


- ▶ The circle $T = \{\text{diag}(e^{i\theta}, e^{-i\theta})\} \cong S^1$
- ▶ Maximal It is not contained in a larger torus of $SU(2)$
- ▶ Recognition Our old diagonal trick was really the maximal torus

Diagonalization, abstractly

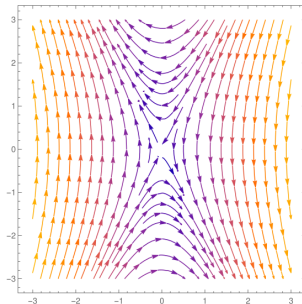
Making axes eigenvectors

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



The xy-axes are eigenvectors

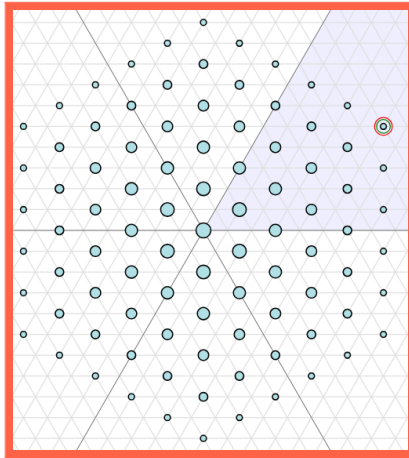
$$\begin{pmatrix} 0 & -1/2 \\ -2 & 0 \end{pmatrix}$$



The xy-axes are not eigenvectors

- **Matrices** Change basis, and a nice matrix becomes diagonal
- **Compact Lie groups** Every $g \in G$ can be conjugated into a maximal torus T
- **Good news** All maximal tori are conjugate: the choice of T does not matter

What does a representation see?



- Restrict to $\mathcal{T}$ We get a commuting family of unitary operators (“rotations”)
- Diagonalize The space splits into simultaneous eigenspaces
- Next question How do we label these eigenspaces?

Thank you for your attention!

Next time: Weights and weight decompositions