Lie theory - part 12

Or: The center, derived algebra, and simple vs solvable

Why these four words matter

$$
\begin{aligned}
Z(X,Y) &= \log(\exp X \exp Y) \\
&= X + Y + \frac{1}{2}[X,Y] + \frac{1}{12}\left([X,[X,Y]] + [Y,[Y,X]]\right) \\
&\quad - \frac{1}{24}[Y,[X,[X,Y]]] \\
&\quad - \frac{1}{720}\left([Y,[Y,[Y,[Y,X]]]] + [X,[X,[X,[X,Y]]]]\right) \\
&\quad + \frac{1}{360}\left([X,[Y,[Y,[Y,X]]]] + [Y,[X,[X,[X,Y]]]]\right) \\
&\quad + \frac{1}{120}\left([Y,[X,[Y,[X,Y]]]] + [X,[Y,[X,[Y,X]]]]\right) \\
&\quad + \frac{1}{240}\left([X,[Y,[X,[Y,[X,Y]]]]]\right) \\
&\quad + \frac{1}{720}\left([X,[Y,[X,[X,[X,Y]]]]] - [X,[X,[Y,[Y,[X,Y]]]]]\right) \\
&\quad + \frac{1}{1440}\left([X,[Y,[Y,[Y,[X,Y]]]]] - [X,[X,[Y,[X,[X,Y]]]]]\right) + \cdots
\end{aligned}
$$

- **Big picture** We now start asking what a Lie algebra is made of
- **First test** How far is our algebra from being commutative, and where does that failure live?
- **Why?** Recall the log-exp picture and how this is easier if commutative

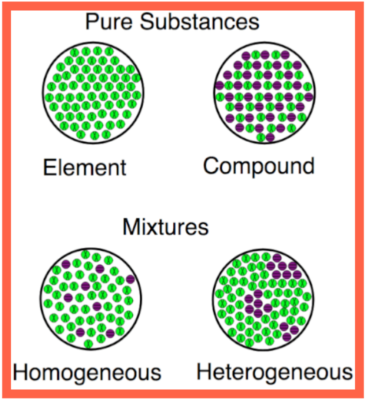
The centre and the derived algebra

- Centre $Z(\mathfrak{g})$ Things that commute with everything, so it measures calmness
- Derived algebra $[\mathfrak{g}, \mathfrak{g}]$ Generated by brackets, so it records the part built from noncommutativity
- Moral One looks for what stays quiet, or what is created by interaction

Two opposite moods: simple and solvable

- Simple Very roughly, there are no meaningful ideals to break off, so the algebra behaves as one stubborn piece
- Solvable Repeated brackets eventually die out, so the algebra becomes more and more calm when probed recursively
- Tension One notion says irreducibly noncommutative, the other says commutativity is hiding not too far away

Why this is a useful dividing line



- Organization These notions split the zoo of examples into families with very different behavior
- Structure theory Much of Lie theory is driven by understanding how complicated objects are assembled from simple pieces
- Examples matter Matrix examples make these words less mysterious

Matrix examples

$$\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \equiv \lambda \boldsymbol{I}_3$$

- **Center** In $\mathfrak{gl}_n$, the scalar matrices commute with everything, so the center is easy to see concretely
- **Derived algebra** For $\mathfrak{gl}_n$, the commutators recover $\mathfrak{sl}_n$, so the derived algebra picks out the genuinely non-abelian part
- **Contrast** Diagonal matrices feel tame and solvable, while $\mathfrak{sl}_n$ points toward the opposite world of simple building blocks

Thank you for your attention!

Next time: Classification of simple Lie algebras