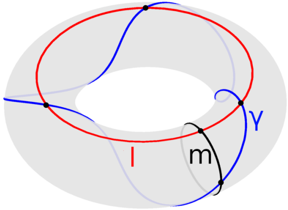
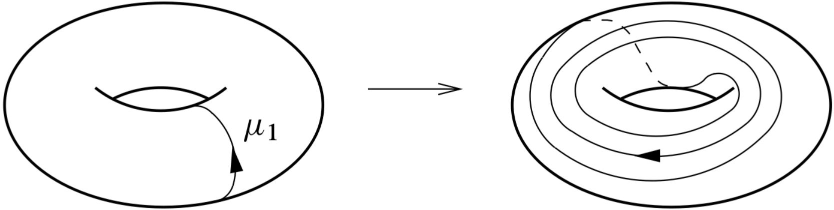
What are...lens spaces?

Or: The birth of geometric topology!?

The meridian winds around



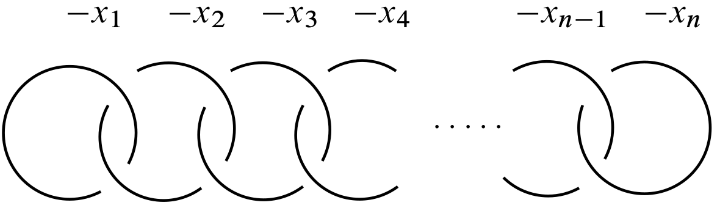
$p = 3$
$q = 1$:



- Dehn surgery Glue the meridian to $[\gamma] = a \cdot [m] + b \cdot [l]$
- The lens space $L(p, q)$, p, q coprime, is determined by $(-q, p)$

Lens spaces and Hopf links

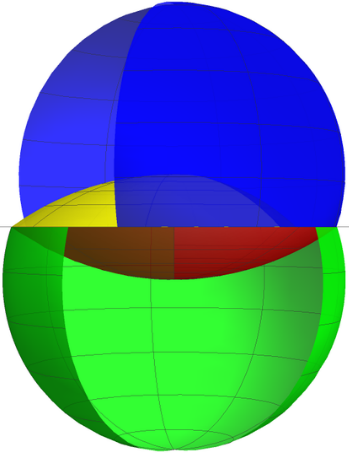
$$[x_1, \ldots, x_n] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{\cdots - \cfrac{1}{x_n}}}$$



- $L(p,q)$ Write $p/q = [x_1, \ldots, x_n]$ as a continued fraction
- The lens space $L(p,q)$ is given by surgery along x_i framed Hopf links
- Example $L(5,2)$; $5/2 = [3,2] = 3 - 1/2 \Rightarrow (-3,-2)$ framed Hopf link

Quotients of spheres

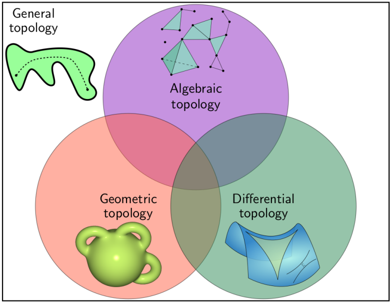
$L(2,5)$:



- Take the map $S^3 \to S^3$ given by $(z_1, z_2) \mapsto (e^{2\pi i/p} \cdot z_1, e^{2\pi iq/p} \cdot z_2)$
- The quotient of S^3 modulo this map is $L(p,q)$
- $L(2,1) \leftrightsquigarrow \mathbb{R}P^3 = S^3/\text{antipodal points}$

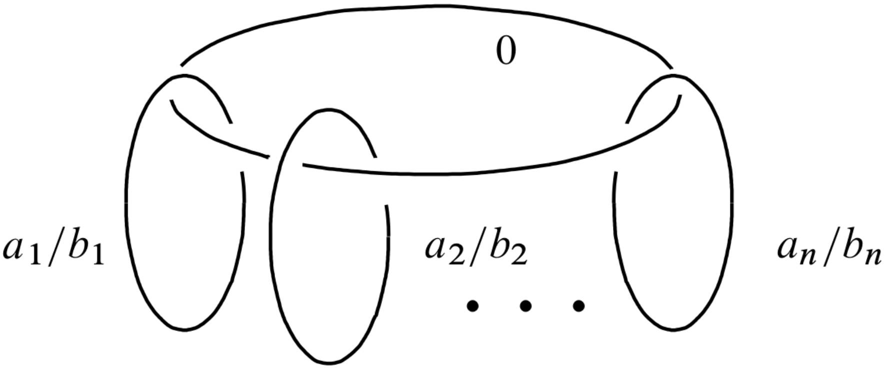
For completeness: A formal statement

The spaces $L(p, q)$ the first known examples of 3mfds which were not determined by their homology and fundamental group alone Birth of geometric topology!?



- $L(5, 1)$ and $L(5, 2)$ are not homeomorphic even though they have isomorphic fundamental groups and the same homology
- $L(7, 1)$ and $L(7, 2)$ are not homeomorphic even though they have the same homotopy type

Seifert manifolds



- Take coprime (a_i, b_i); the Seifert manifold $M(a_1/b_1, \ldots, a_n/b_n)$ is obtained by the above surgery
- Express a_i/b_i as a continued fraction and replace any a_i/b_i framing by a chain of Hopf links as before Generalized Lens spaces
- $\Sigma(p, q, r)$ from the previous video corresponds to $M(p/b_1, q/b_2, r/b_3)$ with $qrb_1 + prb_2 + pqb_3 = 1$

Thank you for your attention!

I hope that was of some help.